15

OSCILLATIONS

Conceptual Questions

15.1. The period of a block oscillating on a spring is $T = 2\pi\sqrt{m/k}$. We are told that $T_1 = 2.0$ s.

(a) In this case the mass is doubled: $m_2 = 2m_1$.

$$\frac{T_2}{T_1} = \frac{2\pi\sqrt{m_2/k}}{2\pi\sqrt{m_1/k}} = \sqrt{\frac{m_2}{m_1}} = \sqrt{\frac{2m_1}{m_1}} = \sqrt{2}$$

So $T_2 = \sqrt{2}T_1 = \sqrt{2}(2.0\text{ s}) = 2.8$ s.

(b) In this case the spring constant is doubled: $k_2 = 2k_1$.

$$\frac{T_2}{T_1} = \frac{2\pi\sqrt{m/k_2}}{2\pi\sqrt{m/k_1}} = \sqrt{\frac{k_1}{k_2}} = \sqrt{\frac{k_1}{2k_1}} = \frac{1}{\sqrt{2}}$$

So $T_2 = T_1/\sqrt{2} = (2.0\text{ s})/\sqrt{2} = 1.4$ s.

(c) The formula for the period does not contain the amplitude; that is, the period is independent of the amplitude. Changing (in particular, doubling) the amplitude does not affect the period, so the new period is still 2.0 s.

It is equally important to understand what *doesn't* appear in a formula. It is quite startling, really, the first time you realize it, that the amplitude doesn't affect the period. But this is crucial to the idea of simple harmonic motion. Of course, if the spring is stretched too far, out of its linear region, then the amplitude would matter.

15.2. The period of a simple pendulum is $T = 2\pi\sqrt{L/g}$. We are told that $T_1 = 2.0$ s.

(a) In this case the mass is doubled: $m_2 = 2m_1$. However, the mass does not appear in the formula for the period of a pendulum; that is, the period does not depend on the mass. Therefore the period is still 2.0 s.

(b) In this case the length is doubled: $L_2 = 2L_1$.

$$\frac{T_2}{T_1} = \frac{2\pi\sqrt{L_2/g}}{2\pi\sqrt{L_1/g}} = \sqrt{\frac{L_2}{L_1}} = \sqrt{\frac{2L_1}{L_1}} = \sqrt{2}$$

So $T_2 = \sqrt{2}T_1 = \sqrt{2}(2.0\text{ s}) = 2.8$ s.

(c) The formula for the period of a simple small-angle pendulum does not contain the amplitude; that is, the period is independent of the amplitude. Changing (in particular, doubling) the amplitude, as long as it is still small, does not affect the period, so the new period is still 2.0 s.

It is equally important to understand what *doesn't* appear in a formula. It is quite startling, really, the first time you realize it, that the amplitude $(\theta_{\max})$ doesn't affect the period. But this is crucial to the idea of simple harmonic motion. Of course, if the pendulum is swung too far, out of its linear region, then the amplitude would matter. The amplitude *does* appear in the formula for a pendulum not restricted to small angles because the small-angle approximation is not valid; but then the motion is not simple harmonic motion.

15.3. **(a)** The amplitude is the maximum displacement obtained. From the graph, $A = 10$ cm.

(b) From the graph, the period $T = 2.0$ s. The angular frequency is thus $\omega = \frac{2\pi}{T} = 3.1$ rad/s.

(c) The phase constant specifies at what point the cosine function starts. From the graph, the starting point looks like $\frac{1}{2}A$. So at $t = 0$, $A\cos\phi = \frac{1}{2}A$ means $\phi_0 = \pm\frac{\pi}{3}(\pm 60°)$. Since the oscillator is moving to the left at $t = 0$, it is in the upper half of the circular-motion diagram, and we choose $\phi_0 = +\frac{\pi}{3}$.

15.4. **(a)** The equilibrium length is 20 cm since that is where the potential energy $U = \frac{1}{2}k(x - x_e)^2 = 0$.

(b) The turning points occur where the potential energy is equal to the total energy. These points are at $x = 14$ cm and 26 cm.

(c) The maximum kinetic energy of 7 J is obtained when the potential energy is minimal, at the equilibrium point.

(d) The total energy will be around 14 J. If a horizontal line is drawn at that energy, it intersects the potential energy curve at the turning points $x = 12$ cm and 28 cm.

15.5. **(a)** A position-vs-time graph plots $x(t) = A\cos(wt + \phi_0)$. The graph of $x(t)$ starts at $\frac{1}{2}A$ and is increasing. So at $t = 0$, $\frac{1}{2}A = A\cos\phi_0 \Rightarrow \phi_0 = \pm\frac{\pi}{3}$. We choose $\phi_0 = -\frac{\pi}{3}$ since the particle is moving to the right, indicating that it is in the bottom half of the circular-motion diagram.

(b) The phase at each point can be determined in the same manner as for part (a). For points 1 and 3, the amplitude is again $\frac{1}{2}A$. At point 1, the particle is moving to the right so $\phi_1 = -\frac{\pi}{3}$. At point 3, the particle is moving left, so $\phi_3 = +\frac{\pi}{3}$. At point 2, the amplitude is A, so $\cos\phi_2 = 1 \Rightarrow \phi_2 = 0$.

15.6. **(a)** A velocity-vs-time graph is a plot of $v(t) = -A\omega\sin(\omega t + \phi_0)$. At $t = 0$, the amplitude is $\frac{1}{2}v_{\max} = \frac{1}{2}A\omega = -A\omega\sin\phi_0$, so $\phi_0 = \sin^{-1}\left(-\frac{1}{2}\right) = \frac{7\pi}{6}$ or $-\frac{\pi}{6}$. Since $v(t = 0 \text{ s})$ is increasing, we need $-\sin(\omega t + \phi_0)$ increasing at $t = 0$, so we choose $\phi_0 = \frac{7\pi}{6}$.

(b) The phase is $\phi = wt + \phi_0$. At points 1 and 3, $v(t) = -\frac{1}{2}v_{\max}$, so $\phi = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. At point 1, we need $-\sin\phi_1$, increasing, so $\phi_1 = \frac{5\pi}{6}$. At point 3, $-\sin\phi_3$ is decreasing, so $\phi_3 = \frac{\pi}{6}$. At point 2, the amplitude is $v_{\max}$, so $-\sin\phi_2 = 1$, thus $\phi_2 = -\frac{\pi}{2}$.

15.7. Energy is transferred back and forth between all potential energy at the extremes $\left(\frac{1}{2}kA^2\right)$ and all kinetic energy at the equilibrium point(s) $\left(\frac{1}{2}mv_{\text{max}}{}^2\right)$. The equation does *not* say that the particle ever has amplitude A and speed v_{max} at the same time. The equation relates expressions for the energy at two different times.

15.8. Because each is circularly symmetric the center of mass is at the center of each. So the length of each pendulum is the same (i.e., R), so they have the same period. The period does not depend on the mass, but even if it did it is also the same in both cases.

15.9. One expression for the total energy is $E = \frac{1}{2}mv_{\text{max}}^2$, which occurs at the equilibrium point when $U = 0$. If the total energy is doubled,

$$2E = \frac{1}{2}m(v'_{\text{max}})^2 = \frac{1}{2}m(\sqrt{2}v_{\text{max}})^2$$

Thus the new maximum speed is $v'_{\text{max}} = \sqrt{2}v_{\text{max}} = \sqrt{2}(20\text{ cm/s}) = 28\text{ cm/s}$

15.10. The time constant $\tau = \frac{m}{b}$ decreases as b increases, meaning energy is removed from the oscillator more quickly since $E(t) = E_0 e^{-t/\tau}$.

(a) The medium is more resistive.

(b) The oscillations damp out more quickly.

(c) The time constant is decreased.

15.11. **(a)** T, the period, is the time for each cycle of the motion, the time required for the motion to repeat itself. τ, the damping time constant, is the time required for the amplitude of a damped oscillator to decrease to $e^{-1} \approx 37\%$ of its original value.

(b) τ, the damping time constant, is the time required for the amplitude of a damped oscillator to decrease to $e^{-1} \approx 37\%$ of its original value, while $t_{1/2}$, the half-life, is the time required for the amplitude of a damped oscillator to decrease to 50% of its original value.

15.12. Natural frequency is the frequency that an oscillator will oscillate at on its own. You may drive an oscillator at a frequency other than its natural frequency. For example, if you are pushing a child in a swing, you build up the amplitude of the oscillation by driving the oscillator at its natural frequency. You can achieve resonance by driving an oscillator at its natural frequency.

Exercises and Problems

Exercises

Section 15.1 Simple Harmonic Motion

15.1. **Model:** The air-track glider oscillating on a spring is in simple harmonic motion.

Solve: The glider completes 10 oscillations in 33 s, and it oscillates between the 10 cm mark and the 60 cm mark.

(a) $T = \dfrac{33\text{ s}}{10\text{ oscillations}} = 3.3\text{ s/oscillation} = 3.3\text{ s}$

(b) $f = \dfrac{1}{T} = \dfrac{1}{3.3\text{ s}} = 0.303\text{ Hz} \approx 0.30\text{ Hz}$

(c) $\omega = 2\pi f = 2\pi(0.303\text{ Hz}) = 1.90\text{ rad/s}$

(d) The oscillation from one side to the other is equal to $60\text{ cm} - 10\text{ cm} = 50\text{ cm} = 0.50\text{ m}$. Thus, the amplitude is $A = \frac{1}{2}(0.50\text{ m}) = 0.25\text{ m}$.

(e) The maximum speed is

$$v_{\max} = \omega A = \left(\frac{2\pi}{T}\right)A = (1.90\text{ rad/s})(0.25\text{ m}) = 0.48\text{ m/s}$$

15.2. Model: The air-track glider attached to a spring is in simple harmonic motion.
Visualize: The position of the glider can be represented as $x(t) = A\cos\omega t$.
Solve: The glider is pulled to the right and released from rest at $t = 0$ s. It then oscillates with a period $T = 2.0$ s and a maximum speed $v_{\max} = 40\text{ cm/s} = 0.40\text{ m/s}$.

(a) $v_{\max} = \omega A$ and $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{2.0\text{ s}} = \pi\text{ rad/s} \Rightarrow A = \dfrac{v_{\max}}{\omega} = \dfrac{0.40\text{ m/s}}{\pi\text{ rad/s}} = 0.127\text{ m} = 12.7\text{ cm} = 13\text{ cm}$

(b) The glider's position at $t = 0.25$ s is

$$x_{0.25\text{ s}} = (0.127\text{ m})\cos[(\pi\text{ rad/s})(0.25\text{ s})] = 0.090\text{ m} = 9.0\text{ cm}$$

15.3. Solve: The frequency generated by a guitar string is 440 Hz. The period is the inverse of the frequency, hence

$$T = \frac{1}{f} = \frac{1}{440\text{ Hz}} = 2.27\times10^{-3}\text{ s} = 2.27\text{ ms}$$

15.4. Solve: The object's position as a function of time is $x(t) = A\cos(\omega t + \phi_0)$. Letting $x = 0$ m at $t = 0$ s, gives

$$0 = A\cos\phi_0 \Rightarrow \phi_0 = \pm\tfrac{1}{2}\pi$$

Since the object is traveling to the right, it is in the lower half of the circular motion diagram, giving a phase constant between $-\pi$ and 0 radians. Thus, $\phi_0 = -\frac{1}{2}\pi$ and

$$x(t) = A\cos(\omega t - \tfrac{1}{2}\pi) \Rightarrow x(t) = A\sin\omega t = (0.10\text{ m})\sin(\tfrac{1}{2}\pi t)$$

where we have used $A = 0.10$ m and

$$\omega = \frac{2\pi}{T} = \frac{2\pi\text{ rad}}{4.0\text{ s}} = \frac{\pi}{2}\text{ rad/s}$$

Let us now find *t* where $x = 0.060$ m:

$$0.060\text{ m} = (0.10\text{ m})\sin\left(\frac{\pi}{2}t\right) \Rightarrow t = \frac{2}{\pi}\sin^{-1}\left(\frac{0.060\text{ m}}{0.10\text{ m}}\right) = 0.41\text{ s}$$

Assess: The answer is reasonable because it is approximately $\frac{1}{8}$ of the period.

Section 15.2 SHM and Circular Motion

15.5. Model: The oscillation is the result of simple harmonic motion.
Solve: (a) The amplitude $A = 20$ cm.
(b) The period $T = 8.0$ s, thus

$$f = \frac{1}{T} = \frac{1}{8.0\text{ s}} = 0.125\text{ Hz} \approx 0.13\text{ Hz}$$

(c) The position of an object undergoing simple harmonic motion is $x(t) = A\cos(\omega t + \phi_0)$. At $t = 0\text{ s}, x_0 = 10\text{ cm}$. Thus,

$$10\text{ cm} = (20\text{ cm})\cos\phi_0 \Rightarrow \phi_0 = \cos^{-1}\left(\frac{10\text{ cm}}{20\text{ cm}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \pm\frac{\pi}{3}\text{ rad} = \pm60°.$$

Because the object is moving to the left at $t = 0$ s, it is in the upper half of the circular motion diagram and thus must have a phase constant between 0 and π radians. Therefore, $\phi_0 = \frac{\pi}{3}$ rad = 60°.

15.6. Model: The oscillation is the result of simple harmonic motion.
Solve: (a) The amplitude $A = 20$ cm.
(b) The time to complete one cycle is the period, hence $T = 8.0$ s and

$$f = \frac{1}{T} = \frac{1}{8.0\text{ s}} = 0.125\text{ Hz} \approx 0.13\text{ Hz}$$

(c) The position of an object undergoing simple harmonic motion is $x(t) = A\cos(\omega t + \phi_0)$. At $t = 0$ s, $x_0 = -10$ cm, thus

$$-10\text{ cm} = (20\text{ cm})\cos[\omega(0\text{ s}) + \phi_0]$$

$$\Rightarrow \cos\phi_0 = \frac{-10\text{ cm}}{20\text{ cm}} = -\frac{1}{2} \Rightarrow \phi_0 = \cos^{-1}\left(-\frac{1}{2}\right) = \pm\frac{2\pi}{3}\text{ rad or } \pm 120°$$

Since the oscillation is originally moving to the left, $\phi_0 = +120°$.

15.7. Solve: The position and the velocity of a particle in simple harmonic motion are

$$x(t) = A\cos(\omega t + \phi_0) \text{ and } v_x(t) = -A\omega\sin(\omega t + \phi_0) = -v_{\text{max}}\sin(\omega t + \phi_0)$$

(a) At $t = 0$ s, the equation for x yields

$$(5.0\text{ cm}) = (10.0\text{ cm})\cos(\phi_0) \Rightarrow \phi_0 = \cos^{-1}(0.5) = \pm\tfrac{2}{3}\pi\text{ rad}$$

Because the particle is moving to the right at $t = 0$ s, it is in the lower half of the circular motion diagram, and the phase constant is between π and 2π radians. Thus, $\phi_0 = -\tfrac{2}{3}\pi$ rad.
(b) The period is 4.0 s. At $t = 0$ s,

$$v_{0x} = -A\omega\sin\phi_0 = -(10.0\text{ cm})\left(\frac{2\pi}{T}\right)\sin\left(-\frac{2\pi}{3}\right) = 13.6\text{ cm/s}$$

(c) The maximum speed is

$$v_{\text{max}} = \omega A = \left(\frac{2\pi}{4.0\text{ s}}\right)(10.0\text{ cm}) = 15.7\text{ cm/s}$$

Assess: The positive velocity at $t = 0$ s is consistent with the position-vs-time graph and the negative sign of the phase constant.

15.8. Solve: The position and the velocity of a particle in simple harmonic motion are

$$x(t) = A\cos(\omega t + \phi_0) \text{ and } v_x(t) = -A\omega\sin(\omega t + \phi_0) = -v_{\text{max}}\sin(\omega t + \phi_0)$$

From the graph, $T = 12$ s and the angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{12\text{ s}} = \frac{\pi}{6}\text{ rad/s}$$

(a) Because $v_{\text{max}} = A\omega = 60$ cm/s, we have

$$A = \frac{60\text{ cm/s}}{\omega} = \frac{60\text{ cm/s}}{\pi/6\text{ rad/s}} = 115\text{ cm}$$

(b) At $t = 0$ s,

$$v_{0x} = -A\omega\sin\phi_0 = 30\text{ cm/s} \Rightarrow -(60\text{ cm/s})\sin\phi_0 = 30\text{ cm/s}$$

$$\Rightarrow \phi_0 = \sin^{-1}(-0.5\text{ rad}) = -\tfrac{1}{6}\pi\text{ rad or } \tfrac{7}{6}\pi\text{ rad}$$

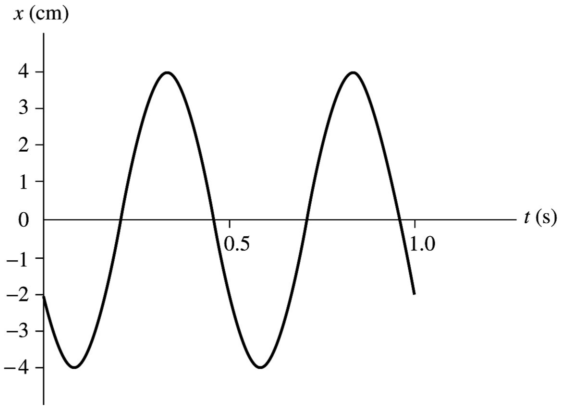
Because the velocity at $t = 0$ s is positive and the particle is slowing down, the particle is in the fourth quadrant of the circular motion diagram. Thus $\phi_0 = -\frac{1}{6}\pi$ rad.

(c) At $t = 0$ s, $x_0 = (115 \text{ cm})\cos(-\frac{1}{6}\pi \text{ rad}) = 100$ cm.

15.9. Visualize: The phase constant $\frac{2}{3}\pi$ has a plus sign, which implies that the object undergoing simple harmonic motion is in the second quadrant of the circular motion diagram. That is, the object is moving to the left.
Solve: The position of the object is

$$x(t) = A\cos(\omega t + \phi_0) = A\cos(2\pi ft + \phi_0) = (4.0 \text{ cm})\cos[(4\pi \text{ rad/s})t + \tfrac{2}{3}\pi \text{ rad}]$$

The amplitude is $A = 4$ cm and the period is $T = 1/f = 0.50$ s. A phase constant $\phi_0 = 2\pi/3 \text{ rad} = 120°$ (second quadrant) means that x starts at $-\frac{1}{2}A$ and is moving to the left (getting more negative).

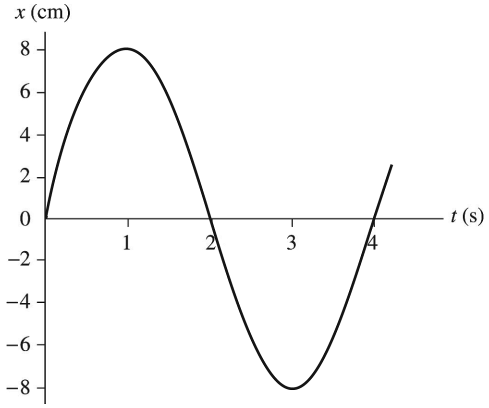


Assess: We can see from the graph that the object starts out moving to the left.

15.10. Visualize: A phase constant of $-\dfrac{\pi}{2}$ implies that the object that undergoes simple harmonic motion is in the lower half of the circular motion diagram. That is, the object is moving to the right.
Solve: The position of the object is given by the equation

$$x(t) = A\cos(\omega t + \phi_0) = A\cos(2\pi ft + \phi_0) = (8.0 \text{ cm})\cos\left[\left(\frac{\pi}{2}\text{ rad/s}\right)t - \frac{\pi}{2}\text{ rad}\right]$$

The amplitude is $A = 8.0$ cm and the period is $T = 1/f = 4.0$ s. With $\phi_0 = -\pi/2$ rad, x starts at 0 cm and is moving to the right (getting more positive).



Assess: As we see from the graph, the object starts out moving to the right.

15.11. Solve: The position of the object is given by the equation

$$x(t) = A\cos(\omega t + \phi_0) = A\cos(2\pi ft + \phi_0)$$

We can find the phase constant ϕ_0 from the initial condition:

$$0 \text{ cm} = (4.0 \text{ cm})\cos\phi_0 \Rightarrow \cos\phi_0 = 0 \Rightarrow \phi_0 = \cos^{-1}(0) = \pm\tfrac{1}{2}\pi \text{ rad}$$

Since the object is moving to the right, the object is in the lower half of the circular motion diagram. Hence, $\phi_0 = -\frac{1}{2}\pi$ rad. The final result, with $f = 4.0$ Hz, is

$$x(t) = (4.0 \text{ cm})\cos[(8.0\pi \text{ rad/s})t - \tfrac{1}{2}\pi \text{ rad}]$$

15.12. Solve: The position of the object is given by the equation

$$x(t) = A\cos(\omega t + \phi_0)$$

The amplitude is $A = 8.0$ cm. The angular frequency $\omega = 2\pi f = 2\pi(0.50 \text{ Hz}) = \pi$ rad/s. Since at $t = 0$ it has its most negative position, it must be about to move to the right, so $\phi_0 = -\pi$. Thus

$$x(t) = (8.0 \text{ cm})\cos[(\pi \text{ rad/s})t - \pi \text{ rad}]$$

15.13. Model: The air-track glider is in simple harmonic motion.
Solve: (a) We can find the phase constant from the initial conditions for position and velocity:

$$x_0 = A\cos\phi_0 \qquad v_{0x} = -\omega A\sin\phi_0$$

Dividing the second by the first, we see that

$$\frac{\sin\phi_0}{\cos\phi_0} = \tan\phi_0 = -\frac{v_{0x}}{\omega x_0}$$

The glider starts to the left ($x_0 = -5.00$ cm) and is moving to the right ($v_{0x} = +36.3$ cm/s). With a period of $1.5 \text{ s} = \frac{3}{2}$ s, the angular frequency is $\omega = 2\pi/T = \frac{4}{3}\pi$ rad/s. Thus

$$\phi_0 = \tan^{-1}\left(-\frac{36.3 \text{ cm/s}}{(4\pi/3 \text{ rad/s})(-5.00 \text{ cm})}\right) = \tfrac{1}{3}\pi \text{ rad } (60°) \text{ or } -\tfrac{2}{3}\pi \text{ rad } (-120°)$$

The tangent function repeats every 180°, so there are always two possible values when evaluating the arctan function. We can distinguish between them because an object with a negative position but moving to the right is in the third quadrant of the corresponding circular motion. Thus $\phi_0 = -\frac{2}{3}\pi$ rad, or $-120°$.

(b) At time *t*, the phase is $\phi = \omega t + \phi_0 = (\frac{4}{3}\pi \text{ rad/s})t - \frac{2}{3}\pi$ rad. This gives $\phi = -\frac{2}{3}\pi$ rad, 0 rad, $\frac{2}{3}\pi$ rad, and $\frac{4}{3}\pi$ rad at, respectively, $t = 0$ s, 0.5 s, 1.0 s, and 1.5 s. This is one period of the motion.

Section 15.3 Energy in SHM

Section 15.4 The Dynamics of SHM

15.14. Model: The block attached to the spring is in simple harmonic motion.
Solve: The period of an object attached to a spring is

$$T = 2\pi\sqrt{\frac{m}{k}} = T_0 = 2.0 \text{ s}$$

where *m* is the mass and *k* is the spring constant.

(a) For mass $= 2m$,

$$T = 2\pi\sqrt{\frac{2m}{k}} = (\sqrt{2})T_0 = 2.8\text{ s}$$

(b) For mass $\frac{1}{2}m$,

$$T = 2\pi\sqrt{\frac{\frac{1}{2}m}{k}} = T_0/\sqrt{2} = 1.41\text{ s}$$

(c) The period is independent of amplitude. Thus $T = T_0 = 2.0\text{ s}$

(d) For a spring constant $= 2k$,

$$T = 2\pi\sqrt{\frac{m}{2k}} = T_0/\sqrt{2} = 1.41\text{ s}$$

15.15. Model: The air-track glider attached to a spring is in simple harmonic motion.
Solve: Experimentally, the period is $T = (12.0\text{ s})/(10\text{ oscillations}) = 1.20\text{ s}$. Using the formula for the period,

$$T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow k = \left(\frac{2\pi}{T}\right)^2 m = \left(\frac{2\pi}{1.20\text{ s}}\right)^2 (0.200\text{ kg}) = 5.48\text{ N/m}$$

15.16. Model: The mass attached to the spring oscillates in simple harmonic motion.
Solve: **(a)** The period $T = 1/f = 1/2.0\text{ Hz} = 0.50\text{ s}$.

(b) The angular frequency $\omega = 2\pi f = 2\pi(2.0\text{ Hz}) = 4\pi\text{ rad/s}$.

(c) Using energy conservation

$$\tfrac{1}{2}kA^2 = \tfrac{1}{2}kx_0^2 + \tfrac{1}{2}mv_{0x}^2$$

Using $x_0 = 5.0\text{ cm}$, $v_{0x} = -30\text{ cm/s}$ and $k = m\omega^2 = (0.200\text{ kg})(4\pi\text{ rad/s})^2$, we get $A = 5.54\text{ cm}$.

(d) To calculate the phase constant ϕ_0,

$$A\cos\phi_0 = x_0 = 5.0\text{ cm}$$
$$\Rightarrow \phi_0 = \cos^{-1}\left(\frac{5.0\text{ cm}}{5.54\text{ cm}}\right) = 0.45\text{ rad}$$

(e) The maximum speed is $v_{\text{max}} = \omega A = (4\pi\text{ rad/s})(5.54\text{ cm}) = 70\text{ cm/s}$.

(f) The maximum acceleration is

$$a_{\text{max}} = \omega^2 A = \omega(\omega A) = (4\pi\text{ rad/s})(70\text{ cm/s}) = 8.8\text{ m/s}^2$$

(g) The total energy is $E = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}(0.200\text{ kg})(0.70\text{ m/s})^2 = 0.049\text{ J}$.

(h) The position at $t = 0.40\text{ s}$ is

$$x_{0.4\text{ s}} = (5.54\text{ cm})\cos[(4\pi\text{ rad/s})(0.40\text{ s}) + 0.45\text{ rad}] = +3.8\text{ cm}$$

15.17. Model: The oscillating mass is in simple harmonic motion.
Solve: **(a)** The amplitude $A = 2.0\text{ cm}$.

(b) The period is calculated as follows:

$$\omega = \frac{2\pi}{T} = 10\text{ rad/s} \Rightarrow T = \frac{2\pi}{10\text{ rad/s}} = 0.63\text{ s}$$

(c) The spring constant is calculated as follows:

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega^2 = (0.050 \text{ kg})(10 \text{ rad/s})^2 = 5.0 \text{ N/m}$$

(d) The phase constant $\phi_0 = -\frac{1}{4}\pi$ rad.

(e) The initial conditions are obtained from the equations

$$x(t) = (2.0 \text{ cm})\cos(10t - \tfrac{1}{4}\pi) \text{ and } v_x(t) = -(20.0 \text{ cm/s})\sin(10t - \tfrac{1}{4}\pi)$$

At $t = 0$ s, these equations become

$$x_0 = (2.0 \text{ cm})\cos(-\tfrac{1}{4}\pi) = 1.41 \text{ cm and } v_{0x} = -(20 \text{ cm/s})\sin(-\tfrac{1}{4}\pi) = 14.1 \text{ cm/s}$$

In other words, the mass is at $+1.41$ cm and moving to the right with a velocity of 14.1 cm/s.

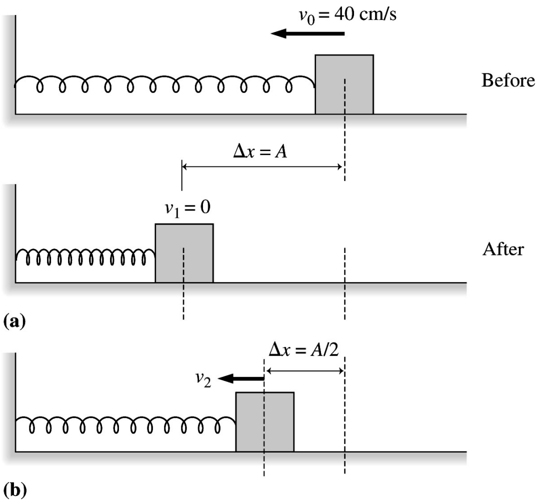
(f) The maximum speed is $v_{\text{max}} = A\omega = (2.0 \text{ cm})(10 \text{ rad/s}) = 20 \text{ cm/s}$.

(g) The total energy $E = \frac{1}{2}kA^2 = \frac{1}{2}(5.0 \text{ N/m})(0.020 \text{ m})^2 = 1.00 \times 10^{-3}$ J.

(h) At $t = 0.41$ s, the velocity is

$$v_{0x} = -(20 \text{ cm/s})\sin[(10 \text{ rad/s})(0.40 \text{ s}) - \tfrac{1}{4}\pi] = 1.46 \text{ cm/s}$$

15.18. Model: The block attached to the spring is in simple harmonic motion.
Visualize:



Solve: **(a)** The conservation of mechanical energy equation $K_f + U_{sf} = K_i + U_{si}$ is

$$\tfrac{1}{2}mv_1^2 + \tfrac{1}{2}k(\Delta x)^2 = \tfrac{1}{2}mv_0^2 + 0 \text{ J} \Rightarrow 0 \text{ J} + \tfrac{1}{2}kA^2 = \tfrac{1}{2}mv_0^2 + 0 \text{ J}$$

$$\Rightarrow A = \sqrt{\frac{m}{k}}v_0 = \sqrt{\frac{1.0 \text{ kg}}{16 \text{ N/m}}}(0.40 \text{ m/s}) = 0.10 \text{ m} = 10 \text{ cm}$$

(b) We have to find the velocity at a point where $x = A/2$. The conservation of mechanical energy equation $K_2 + U_{s2} = K_i + U_{si}$ is

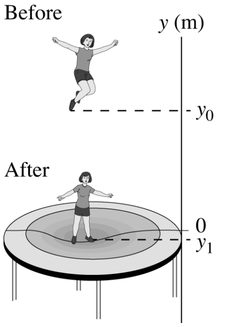
$$\frac{1}{2}mv_2^2 + \frac{1}{2}k\left(\frac{A}{2}\right)^2 = \frac{1}{2}mv_0^2 + 0 \text{ J} \Rightarrow \frac{1}{2}mv_2^2 = \frac{1}{2}mv_0^2 - \frac{1}{4}\left(\frac{1}{2}kA^2\right) = \frac{1}{2}mv_0^2 - \frac{1}{4}\left(\frac{1}{2}mv_0^2\right) = \frac{3}{4}\left(\frac{1}{2}mv_0^2\right)$$

$$\Rightarrow v_2 = \sqrt{\frac{3}{4}}v_0 = \sqrt{\frac{3}{4}}(0.40 \text{ m/s}) = 0.346 \text{ m/s}$$

The velocity is 35 cm/s.

15.19. Model: She is free fall when not in contact with the trampoline. SHM while she is in contact.

Visualize: Use conservation of energy, with gravitational potential energy transformed into spring potential energy.



Known
$y_0 = 0.55$ m
$y_1 = -0.15$ m

Find
½ T

Solve: $E_\text{i} = E_\text{f}$

$$mgy_0 = \tfrac{1}{2}ky_1^2 + mgy_1$$

Not that y_1 is negative and $|y_1| = A$. Divide by m and combine terms.

$$2g(y_0 - y_1) = \frac{k}{m}y_1^2$$

Note that $\omega^2 = k/m$ and recall $\omega = 2\pi/T$.

$$\omega^2 = \frac{2g(y_0 - y_1)}{y_1^2} \Rightarrow \omega = \sqrt{\frac{2g(y_0 - y_1)}{y_1^2}} = \frac{2\pi}{T} \Rightarrow$$

$$T = 2\pi\sqrt{\frac{y_1^2}{2g(y_0 - y_1)}} = 2\pi\sqrt{\frac{(-0.15\text{ m})^2}{2(9.8\text{ m/s}^2)(0.55\text{ m} - (-0.15\text{ m}))}} = 0.254\text{ s}$$

The time in contact with the trampoline is only half a period (during the stretch down and back to equilibrium, but not up); so the time in contact is 0.13 s.

Assess: The contact time is short because she was moving fast when she first touched the trampoline.

15.20. Solve: From the figure, the amplitude of the oscillation is $A = x_\text{max} = 2$ m. We also see that $K_\text{max} = 8$ J.

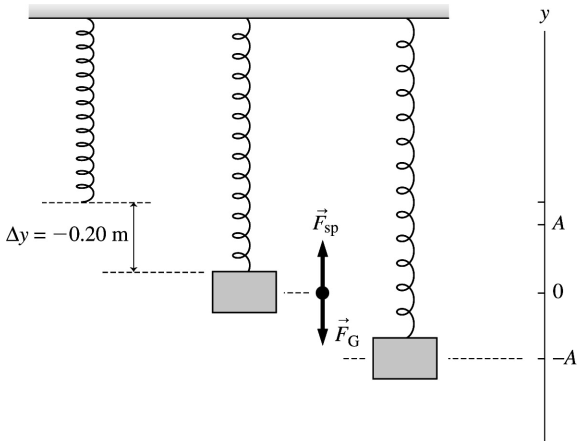
$$E_\text{tot} = K_\text{max} = U_\text{max} = \tfrac{1}{2}kA^2 \Rightarrow k = \frac{2K_\text{max}}{A^2} = \frac{2(8\text{ J})}{(2\text{ m})^2} = 4\text{ N/m}$$

Assess: This is a possible value of k for a long spring, and the units check out.

Section 15.5 Vertical Oscillations

15.21. Model: The vertical oscillations constitute simple harmonic motion.

Visualize:



Solve: **(a)** At equilibrium, Newton's first law applied to the physics book is

$$(F_{sp})_y - mg = 0 \text{ N} \Rightarrow -k\Delta y - mg = 0 \text{ N}$$

$$\Rightarrow k = -mg/\Delta y = -(0.500 \text{ kg})(9.8 \text{ m/s}^2)/(-0.20 \text{ m}) = 24.5 \text{ N/m} \approx 25 \text{ N/m}$$

(b) To calculate the period:

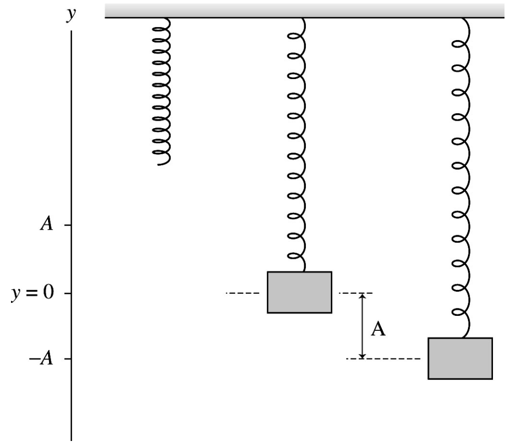
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{24.5 \text{ N/m}}{0.500 \text{ kg}}} = 7.0 \text{ rad/s and } T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{7.0 \text{ rad/s}} = 0.90 \text{ s}$$

(c) The maximum speed is

$$v_{max} = A\omega = (0.10 \text{ m})(7.0 \text{ rad/s}) = 0.70 \text{ m/s}$$

Maximum speed occurs as the book passes through the equilibrium position.

15.22. **Model:** The vertical oscillations constitute simple harmonic motion.
Visualize:



Solve: The period and angular frequency are

$$T = \frac{20 \text{ s}}{30 \text{ oscillations}} = 0.6667 \text{ s and } \omega = \frac{2\pi}{T} = \frac{2\pi}{0.6667 \text{ s}} = 9.425 \text{ rad/s}$$

(a) The mass can be found as follows:

$$\omega=\sqrt{\frac{k}{m}}\Rightarrow m=\frac{k}{\omega^2}=\frac{15\text{ N/m}}{(9.425\text{ rad/s})^2}=0.169\text{ kg}\approx 0.17\text{ kg}$$

(b) The maximum speed $v_{\max}=\omega A=(9.425\text{ rad/s})(0.060\text{ m})=0.57\text{ m/s}$.

15.23. Model: The vertical oscillations constitute simple harmonic motion.

Solve: To find the oscillation frequency using $\omega=2\pi f=\sqrt{k/m}$, we first need to find the spring constant *k*. In equilibrium, the weight mg of the block and the spring force $k\Delta L$ are equal and opposite. That is, $mg=k\Delta L\Rightarrow k=mg/\Delta L$. The frequency of oscillation *f* is thus given as

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}=\frac{1}{2\pi}\sqrt{\frac{mg/\Delta L}{m}}=\frac{1}{2\pi}\sqrt{\frac{g}{\Delta L}}=\frac{1}{2\pi}\sqrt{\frac{9.8\text{ m/s}^2}{0.020\text{ m}}}=3.5\text{ Hz}$$

Section 15.6 The Pendulum

15.24. Model: Model the clock as a simple pendulum undergoing SHM.

Visualize: The pendulum passes through the lowest point twice each period, so $T=2.0$ s.

Solve:

$$T=2\pi\sqrt{\frac{L}{g}}\Rightarrow L=\frac{T^2g}{4\pi^2}=\frac{(2.0\text{ s})^2(9.8\text{ m/s}^2)}{4\pi^2}=0.99\text{ m}$$

Assess: This is approximately the length of the pendulum in my grandfather's clock; it may be a little different because the vertical rod to the bob is not massless. The bob can be moved up and down to allow for changes in length due to temperature variation.

15.25. Model: Assume the small-angle approximation so there is simple harmonic motion.

Solve: The period is $T=12\text{ s}/10\text{ oscillations}=1.20\text{ s}$ and is given by the formula

$$T=2\pi\sqrt{\frac{L}{g}}\Rightarrow L=\left(\frac{T}{2\pi}\right)^2g=\left(\frac{1.20\text{ s}}{2\pi}\right)^2(9.8\text{ m/s}^2)=36\text{ cm}$$

15.26. Model: Assume a small angle of oscillation so there is simple harmonic motion.

Solve: The period of the pendulum is

$$T_0=2\pi\sqrt{\frac{L_0}{g}}=4.0\text{ s}$$

(a) The period is independent of the mass and depends only on the length. Thus $T=T_0=4.0$ s.

(b) For a new length $L=2L_0$,

$$T=2\pi\sqrt{\frac{2L_0}{g}}=\sqrt{2}T_0=5.7\text{ s}$$

(c) For a new length $L=L_0/2$,

$$T=2\pi\sqrt{\frac{L_0/2}{g}}=\frac{1}{\sqrt{2}}T_0=2.8\text{ s}$$

(d) The period is independent of the amplitude as long as there is simple harmonic motion. Thus $T=4.0$ s.

15.27. Model: Assume the pendulum to have small-angle oscillations. In this case, the pendulum undergoes simple harmonic motion.

Solve: Using the formula $g=GM/R^2$, the periods of the pendulums on the moon and on the earth are

$$T_{\text{earth}} = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{L_{\text{earth}}R_{\text{earth}}^2}{GM_{\text{earth}}}} \text{ and } T_{\text{moon}} = 2\pi\sqrt{\frac{L_{\text{moon}}R_{\text{moon}}^2}{GM_{\text{moon}}}}$$

Because $T_{\text{earth}} = T_{\text{moon}}$,

$$2\pi\sqrt{\frac{L_{\text{earth}}R_{\text{earth}}^2}{GM_{\text{earth}}}} = 2\pi\sqrt{\frac{L_{\text{moon}}R_{\text{moon}}^2}{GM_{\text{moon}}}} \Rightarrow L_{\text{moon}} = \left(\frac{M_{\text{moon}}}{M_{\text{earth}}}\right)\left(\frac{R_{\text{earth}}}{R_{\text{moon}}}\right)^2 L_{\text{earth}}$$

$$= \left(\frac{7.36\times10^{22}\text{ kg}}{5.98\times10^{24}\text{ kg}}\right)\left(\frac{6.37\times10^6\text{ m}}{1.74\times10^6\text{ m}}\right)^2 (2.0\text{ m}) = 33\text{ cm}$$

15.28. Model: Assume a small angle of oscillation so there is simple harmonic motion.
Solve: (a) On the earth the period is

$$T_{\text{earth}} = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.0\text{ m}}{9.80\text{ m/s}^2}} = 2.0\text{ s}$$

(b) On Venus the acceleration due to gravity is

$$g_{\text{Venus}} = \frac{GM_{\text{Venus}}}{R_{\text{Venus}}^2} = \frac{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(4.88\times10^{24}\text{ kg})}{(6.06\times10^6\text{ m})^2} = 8.86\text{ m/s}^2$$

$$\Rightarrow T_{\text{Venus}} = 2\pi\sqrt{\frac{L}{g_{\text{Venus}}}} = 2\pi\sqrt{\frac{1.0\text{ m}}{8.86\text{ m/s}^2}} = 2.1\text{ s}$$

15.29. Model: Assume a small angle of oscillation so that the pendulum has simple harmonic motion.
Solve: The time periods of the pendulums on the earth and on Mars are

$$T_{\text{earth}} = 2\pi\sqrt{\frac{L}{g_{\text{earth}}}} \text{ and } T_{\text{Mars}} = 2\pi\sqrt{\frac{L}{g_{\text{Mars}}}}$$

Dividing these two equations,

$$\frac{T_{\text{earth}}}{T_{\text{Mars}}} = \sqrt{\frac{g_{\text{Mars}}}{g_{\text{earth}}}} \Rightarrow g_{\text{Mars}} = g_{\text{earth}}\left(\frac{T_{\text{earth}}}{T_{\text{Mars}}}\right)^2 = (9.8\text{ m/s}^2)\left(\frac{1.50\text{ s}}{2.45\text{ s}}\right)^2 = 3.67\text{ m/s}^2$$

15.30. Model: Assume that the angle with the vertical that the pendulum makes is small enough so that there is simple harmonic motion.
Solve: The angle θ made by the string with the vertical as a function of time is

$$\theta(t) = \theta_{\max}\cos(\omega t + \phi_0)$$

The pendulum starts from maximum displacement, thus $\phi_0 = 0$. Thus, $\theta(t) = \theta_{\max}\cos\omega t$. To find the time t when the pendulum reaches 4.0° on the opposite side:

$$(-4.0°) = (8.0°)\cos\omega t \Rightarrow \omega t = \cos^{-1}(-0.5) = 2.094\text{ rad}$$

Using the formula for the angular frequency,

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8\text{ m/s}^2}{1.0\text{ m}}} = 3.130\text{ rad/s} \Rightarrow t = \frac{2.0944\text{ rad}}{\omega} = \frac{2.094\text{ rad}}{3.130\text{ rad/s}} = 0.669\text{ s}$$

The time $t = 0.67$ s.
Assess: Because $T = 2\pi/\omega = 2.0$ s, a value of 0.67 s for the pendulum to cover a little less than half the oscillation is reasonable.

15.31. Model: This is not a simple pendulum, but is a physical pendulum. Model the bar as thin enough to use $I = \frac{1}{3}ML^2$ where L is the full length of the rod.

Visualize: For a small-angle physical pendulum $T = 2\pi\sqrt{\frac{I}{Mgl}}$ where l is the distance from the pivot to the center of mass. Note that $l = L/2$. We seek L.
Solve: Substitute in for I and then solve for L.

$$T = 2\pi\sqrt{\frac{I}{Mgl}} = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{MgL/2}} = 2\pi\sqrt{\frac{\frac{2}{3}L^2}{gL}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$L = \left(\frac{T}{2\pi}\right)^2\frac{3g}{2} = \frac{3}{8}\frac{g}{\pi^2}T^2 = \frac{3}{8}\frac{(9.8\text{ m/s}^2)}{\pi^2}(1.2\text{ s})^2 = 0.54\text{ m} = 54\text{ cm}$$

Assess: It seems reasonable that a uniform bar 54 cm long would have a period of 1.2 s.

Section 15.7 Damped Oscillations

Section 15.8 Driven Oscillations and Resonance

15.32. Model: The spider is in simple harmonic motion.
Solve: Your tapping is a driving frequency. Largest amplitude at $f_{\text{ext}} = 1.0$ Hz means that this is the resonance frequency, so $f_0 = f_{\text{ext}} = 1.0$ Hz. That is, the spider's natural frequency of oscillation f_0 is 1.0 Hz and $\omega_0 = 2\pi f_0 = 2\pi$ rad/s. We have

$$\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_0^2 = (0.0020\text{ kg})(2\pi\text{ rad/s})^2 = 0.079\text{ N/m}$$

15.33. Model: The motion is a damped oscillation.
Solve: The amplitude of the oscillation at time t is given by the equation $A(t) = A_0e^{-t/2\tau}$, where $\tau = m/b$ is the time constant. Using $x = 0.368\text{ A}$ and $t = 10.0\text{ s}$, we get

$$0.368A = Ae^{-10.0\text{ s}/2\tau} \Rightarrow \ln(0.368) = \frac{-10\text{ s}}{2\tau} \Rightarrow \tau = -\frac{10.0\text{ s}}{2\ln(0.368)} = 5.00\text{ s}$$

15.34. Model: The pendulum is a damped oscillator.
Solve: The period of the pendulum and the number of oscillations in 4 hours are calculated as follows:

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{15.0\text{ m}}{9.8\text{ m/s}^2}} = 7.773\text{ s} \Rightarrow N_{\text{osc}} = \frac{4(3600\text{ s})}{7.773\text{ s}} = 1850$$

The amplitude of the pendulum as a function of time is $A(t) = Ae^{-bt/2m}$. The exponent of this expression can be calculated to be

$$-\frac{bt}{2m} = -\frac{(0.010\text{ kg/s})(4\times3600\text{ s})}{2(110\text{ kg})} = -0.6545$$

We have $A(t) = (1.50\text{ m})e^{-0.6545} = 0.78$ m.

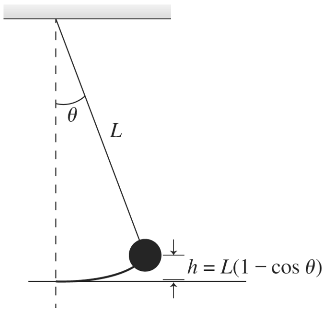
15.35. Model: Assume the eye is a simple driven oscillator.
Visualize: Given the mass and the resonant frequency, we can determine the effective spring constant using the relationship $\omega = 2\pi f = \sqrt{k/m}$.
Solve: Solving the above expression for the spring constant, obtain

$$k = (2\pi f)^2 m = [2\pi(29\text{ Hz})]^2(7.5\times10^{-3}\text{ kg}) = 249\text{ N/m} \approx 250\text{ N/m}$$

Assess: As spring constants go, this is a fairly large value, however the musculature holding the eyeball in the socket is strong and hence will have a large effective spring constant.

15.36. Model: Model the string as massless.

Visualize:



Known
$b = 0.010$ kg/s
$m = 0.35$ kg
$L = 0.45$ m
$\theta = 4.5°$
$t = 25$ s

Find
E_L

Solve: **(a)** First we find the period of the damped oscillations.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \dfrac{b^2}{4m^2}}} = \frac{2\pi}{\sqrt{\dfrac{g}{L} - \dfrac{b^2}{4m^2}}} = \frac{2\pi}{\sqrt{\dfrac{9.8\text{ m/s}^2}{0.45\text{ m}} - \dfrac{(0.010\text{ kg/s})^2}{4(0.35\text{ kg})^2}}} = 1.346\text{ s}$$

The number of oscillations in 25 s will be

$$\frac{25\text{ s}}{T} = \frac{25\text{ s}}{1.346\text{ s}} = 18.6$$

So the oscillator has completed 18 oscillations.
(b) Examine the figure to see that

$$E_0 = mgh = mgL(1 - \cos\theta)$$

The energy at a later time is $E(t) = E_0 e^{-t/\tau} = E_0 e^{-bt/m}$. The energy lost E_L is

$$\begin{aligned} E_L &= E_0 - E = E_0(1 - e^{-bt/m}) = mgL(1 - \cos\theta)(1 - e^{-bt/m}) \\ &= (0.35\text{ kg})(9.8\text{ m/s}^2)(0.45\text{ m})(1 - \cos 4.5°)(1 - e^{-(0.010\text{ kg/s})(25\text{ s})/(0.35\text{ kg})}) = 2.4\text{ mJ} \end{aligned}$$

Assess: Not much energy was lost because the damping constant was small.

Problems

15.37. Model: The particle is in simple harmonic motion.
Solve: The equation for the velocity of the particle is

$$v_x(t) = -(25\text{ cm})(10\text{ rad/s})\sin(10\,t)$$

Substituting into $K = 2U$ gives

$$\begin{aligned} \frac{1}{2}mv_x^2(t) = 2\left(\frac{1}{2}kx^2(t)\right) &\Rightarrow \frac{1}{2}m[-(250\text{ cm/s})\sin(10\,t)]^2 = k[(25\text{ cm})\cos(10\,t)]^2 \\ &\Rightarrow \frac{\sin^2(10\,t)}{\cos^2(10\,t)} = 2\left(\frac{k}{m}\right)\frac{(25\text{ cm})^2}{(250\text{ cm/s})^2} = 2\omega^2\left(\frac{1}{100}\right)\text{s}^2 \\ &\Rightarrow \tan^2(10\,t) = 2(10\text{ rad/s})^2\left(\frac{1}{100}\right)\text{s}^2 = 2.0 \Rightarrow t = \frac{1}{10}\tan^{-1}\sqrt{2.0} = 0.096\text{ s} \end{aligned}$$

15.38. Model: The spring undergoes simple harmonic motion.

Solve: **(a)** Total energy is $E = \frac{1}{2}kA^2$. When the displacement is $x = \frac{1}{2}A$, the potential energy is

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}k\left(\tfrac{1}{2}A\right)^2 = \tfrac{1}{4}\left(\tfrac{1}{2}kA^2\right) = \tfrac{1}{4}E \Rightarrow K = E - U = \tfrac{3}{4}E$$

One quarter of the energy is potential and three-quarters is kinetic.

(b) To have $U = \tfrac{1}{2}E$ requires

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}E = \tfrac{1}{2}\left(\tfrac{1}{2}kA^2\right) \Rightarrow x = \frac{A}{\sqrt{2}}$$

15.39. Solve: Average speed is $v_{\text{avg}} = \Delta x/\Delta t$. During half a period $(\Delta t = \tfrac{1}{2}T)$, the particle moves from $x = -A$ to $x = +A(\Delta x = 2A)$. Thus

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{2A}{T/2} = \frac{4A}{T} = \frac{4A}{2\pi/\omega} = \frac{2}{\pi}(\omega A) = \frac{2}{\pi}v_{\max} \Rightarrow v_{\max} = \frac{\pi}{2}v_{\text{avg}}$$

15.40. Model: The ball attached to a spring is in simple harmonic motion.

Solve: (a) Let $t = 0$ s be the instant when $x_0 = -5.0$ cm and $v_0 = 20$ cm/s. The oscillation frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.5\ \text{N/m}}{0.100\ \text{kg}}} = 5.0\ \text{rad/s}$$

Solving Equation 15.26 for the amplitude gives

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{(-5.0\ \text{cm})^2 + \left(\frac{20\ \text{cm/s}}{5.0\ \text{rad/s}}\right)^2} = 6.4\ \text{cm}$$

(b) The maximum acceleration is $a_{\max} = \omega^2 A = 160\,\text{cm/s}^2$.

(c) For an oscillator, the acceleration is most positive $(a = a_{\max})$ when the displacement is most negative $(x = -x_{\max} = -A)$. So the acceleration is maximum when $x = -6.4$ cm.

(d) We can use the conservation of energy between $x_0 = -5.0$ cm and $x_1 = 3.0$ cm:

$$\tfrac{1}{2}mv_0^2 + \tfrac{1}{2}kx_0^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 \Rightarrow v_1 = \sqrt{v_0^2 + \frac{k}{m}(x_0^2 - x_1^2)} = 0.283\,\text{m/s}$$

The speed is 28 cm/s. Because *k* is known in SI units of N/m, the energy calculation *must* be done using SI units of m, m/s, and kg.

15.41. Model: The oscillator is in simple harmonic motion. Energy is conserved.

Solve: The energy conservation equation $E_1 = E_2$ is

$$\tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2$$

$$\frac{1}{2}(0.30\ \text{kg})(0.954\ \text{m/s})^2 + \frac{1}{2}k(0.030\ \text{m})^2 = \frac{1}{2}(0.30\ \text{kg})(0.714\ \text{m/s})^2 + \frac{1}{2}k(0.060\ \text{m})^2$$

$$\Rightarrow k = 44.48\,\text{N/m}$$

The total energy of the oscillator is

$$E_{\text{total}} = \frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}(0.30\ \text{kg})(0.954\ \text{m/s})^2 + \frac{1}{2}(44.48\ \text{N/m})(0.030\ \text{m})^2 = 0.1565\ \text{J}$$

Because $E_{\text{total}} = \tfrac{1}{2}mv_{\max}^2$,

$$0.1565\ \text{J} = \frac{1}{2}(0.300\ \text{kg})v_{\max}^2 \Rightarrow v_{\ \max} = 1.02\,\text{m/s}$$

Assess: A maximum speed of 1.02 m/s is reasonable.

15.42. Model: The transducer undergoes simple harmonic motion.
Solve: Newton's second law for the transducer is

$$F_{\text{restoring}} = ma_{\text{max}} \Rightarrow 40{,}000\text{ N} = (0.10\times10^{-3}\text{ kg})a_{\text{max}} \Rightarrow a_{\text{max}} = 4.0\times10^{8}\text{ m/s}^2$$

Because $a_{\text{max}} = \omega^2 A$,

$$A = \frac{a_{\text{max}}}{\omega^2} = \frac{4.0\times10^{8}\text{ m/s}^2}{[2\pi(1.0\times10^{6}\text{ Hz})]^2} = 1.01\times10^{-5}\text{ m} = 10.1\ \mu\text{m}$$

(b) The maximum velocity is

$$v_{\text{max}} = \omega A = 2\pi(1.0\times10^{6}\text{ Hz})(1.01\times10^{-5}\text{ m}) = 64\text{ m/s}$$

15.43. Model: The astronaut attached to the spring is in simple harmonic motion.
Solve: (a) From the graph, $T = 3.0$ s, so we have

$$T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow m = \left(\frac{T}{2\pi}\right)^2 k = \left(\frac{3.0\text{ s}}{2\pi}\right)^2 (240\text{ N/m}) = 55\text{ kg}$$

(b) Oscillations occur about an equilibrium position of 1.0 m. From the graph, $A = \frac{1}{2}(0.80\text{ m}) = 0.40\text{ m}$, $\phi_0 = 0$ rad, and

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.0\text{ s}} = 2.1\text{ rad/s}$$

The equation for the position of the astronaut is

$$x(t) = A\cos\omega t + 1.0\text{ m} = (0.4\text{ m})\cos[(2.1\text{ rad/s})t] + 1.0\text{ m}$$
$$\Rightarrow 1.2\text{ m} = (0.4\text{ m})\cos[(2.1\text{ rad/s})t] + 1.0\text{ m} \Rightarrow \cos[(2.1\text{ rad/s})t] = 0.5 \Rightarrow t = 0.50\text{ s}$$

The equation for the velocity of the astronaut is

$$v_x(t) = -A\omega\sin(\omega t)$$
$$\Rightarrow v_{0.5\text{ s}} = -(0.4\text{ m})(2.1\text{ rad/s})\sin[(2.1\text{ rad/s})(0.50\text{ s})] = -0.73\text{ m/s}$$

Thus her speed is 0.73 m/s.

15.44. Model: The block attached to the spring is in simple harmonic motion.
Solve: (a) The frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2000\text{ N/m}}{5.0\text{ kg}}} = 3.183\text{ Hz}$$

The frequency is 3.2 Hz.
(b) From energy conservation,

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{(0.050\text{ m})^2 + \left(\frac{1.0\text{ m/s}}{2\pi\cdot 3.183\text{ Hz}}\right)^2} = 0.0707\text{ m}$$

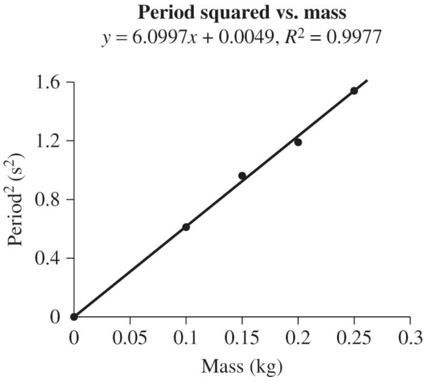
The amplitude is 7.1 cm.
(c) The total mechanical energy is

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}(2000\text{ N/m})(0.0707\text{ m})^2 = 5.0\text{ J}$$

15.45. Model: Model this situation as a simple harmonic oscillator without damping.
Visualize: The period is independent of the amplitude. If the oscillator is undamped the amplitude will be constant; the amplitude is just the arbitrary distance from the equilibrium from which the mass was released. The column of amplitude data is not needed.

The period is related to the spring constant however: $T = 2\pi\sqrt{\frac{m}{k}}$.

Solve: The equation $T^2 = 4\pi^2 \frac{m}{k}$ leads us to believe that a graph of T^2 vs. m would give a straight line graph whose slope is $4\pi^2/k$. Note that the period is 1/10 of the times given in the table.

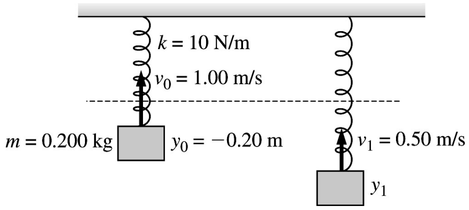


From the spreadsheet we see that the linear fit is very good and that the slope is $6.0997\text{s}^2/\text{kg}$.

$$\text{slope} = 4\pi^2/k \Rightarrow k = \frac{4\pi^2}{6.0997\text{s}^2/\text{kg}} = 6.5\text{ N/m}$$

Assess: Check the units carefully in the last equation. 6.5 N/m is a reasonable spring constant.

15.46. Model: The block undergoes simple harmonic motion.
Visualize:



Solve: **(a)** The frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{10\text{ N/m}}{0.20\text{ kg}}} = 1.125\text{ Hz}$$

The frequency is 1.1 Hz.

(b) Using conservation of energy, $\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$, we find

$$x_1 = \sqrt{x_0^2 + \frac{m}{k}(v_0^2 - v_1^2)} = \sqrt{(-0.20\text{ m})^2 + \frac{0.20\text{ kg}}{10\text{ N/m}}((1.00\text{ m/s})^2 - (0.50\text{ m/s})^2)}$$
$$= 0.2345\text{ m or 23 cm}$$

(c) At time t, the displacement is $x = A\cos(\omega t + \phi_0)$. The angular frequency is $\omega = 2\pi f = 7.071$ rad/s. The amplitude is

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{(-0.20\text{ m})^2 + \left(\frac{1.00\text{ m/s}}{7.071\text{ rad/s}}\right)^2} = 0.245\text{ m}$$

The phase constant is

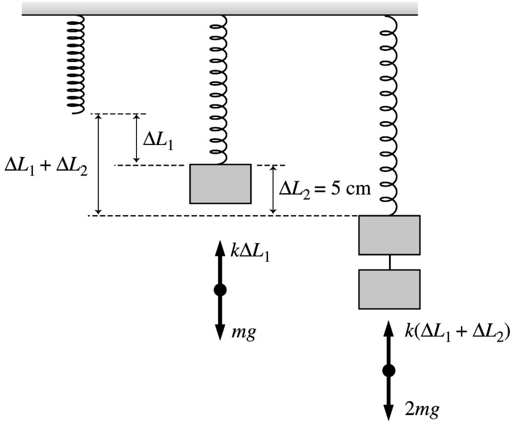
$$\phi_0 = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}\left(\frac{-0.200\text{ m}}{0.245\text{ m}}\right) = \pm 2.526\text{ rad or } \pm 145°$$

A negative displacement (below the equilibrium point) and positive velocity (upward motion) indicate that the corresponding circular motion is in the third quadrant, so $\phi_0 = -2.526$ rad. Thus at $t = 1.0$ s,

$$x = (0.245\text{ m})\cos((7.071\text{ rad/s})(1.0\text{ s}) - 2.526\text{ rad}) = -0.0409\text{ m} = -4.09\text{ cm}$$

The block is 4.1 cm below the equilibrium point.

15.47. Model: The blocks undergo simple harmonic motion.
Visualize:



The length of the stretched spring due to a block of mass m is ΔL_1. In the case of the two-block system, the spring is further stretched by an amount ΔL_2.

Solve: The equilibrium equations from Newton's second law for the single-block and double-block systems are

$$(\Delta L_1)k = mg \text{ and } (\Delta L_1 + \Delta L_2)k = (2m)g$$

Using $\Delta L_2 = 5.0$ cm, and subtracting these two equations, gives us

$$(\Delta L_1 + \Delta L_2)k - \Delta L_1 k = (2m)g - mg \Rightarrow (0.05\text{ m})k = mg$$

$$\Rightarrow \frac{k}{m} = \frac{9.8\text{ m/s}^2}{0.05\text{ m}} = 196\text{ s}^{-2}$$

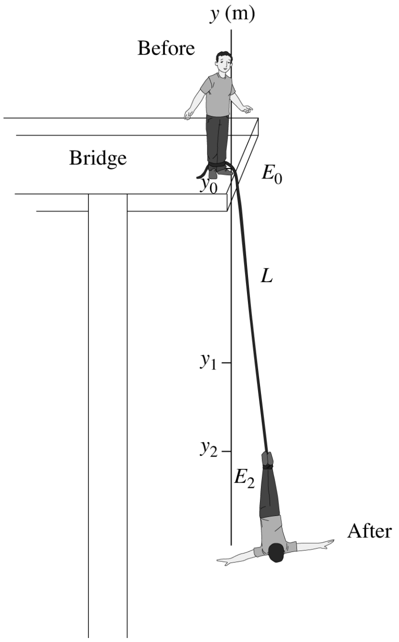
With both blocks attached, giving total mass $2m$, the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{2m}} = \sqrt{\frac{1}{2}\frac{k}{m}} = \sqrt{\frac{1}{2}196\text{ s}^{-2}} = 9.90\text{ rad/s}$$

Thus the oscillation frequency is $f = \omega/2\pi = 1.6$ Hz.

15.48. Model: Model the bungee cord as massless with a Hooke's law restoring force.

Visualize:



Known
$y_0 = 0$
$y_1 = -L = -12$ m
$m = 75$ kg
$k = 430$ n/m

Find
y_2
Time to reach y_2

Solve: **(a)** Use energy conservation where $E_0 = 0$ at the top of the bridge. The energy at the lowest point is $E_2 = \frac{1}{2}k(y_2 - y_1)^2 + mgy_2$ where $y_1 = -L$ and $y_2 - y_1$ is the maximum stretch of the bungee and both y_1 and y_2 are negative. We seek y_2.

$$E_0 = E_2 \Rightarrow \frac{1}{2}k(y_2 - y_1)^2 + mgy_2 = 0$$

Square the binomial, multiply through by 2/*k*, and collect terms to get this quadratic equation:

$$y_2^2 + \left(2\frac{m}{k}g - 2y_1\right)y_2 + y_1^2 = 0$$

Use the quadratic formula.

$$y_2 = \frac{-\left(2\frac{m}{k}g - 2y_1\right) \pm \sqrt{\left(2\frac{m}{k}g - 2y_1\right)^2 - 4y_1^2}}{2}$$

Cancel a 2 from every term in top and bottom.

$$\begin{aligned} y_2 &= -\left(\frac{m}{k}g - y_1\right) \pm \sqrt{\left(\frac{m}{k}g - y_1\right) - y_1^2} \\ &= -\left(\frac{(75\text{ kg})}{(430\text{ N/m})}9.8\text{ m/s}^2 - (-12\text{ m})\right)^2 \pm \sqrt{\left(\frac{(75\text{ kg})}{(430\text{ N/m})}9.8\text{ m/s}^2 - (-12\text{ m})\right)^2 - (12\text{ m})^2} \\ &= -7.1\text{ m or } -20\text{ m} \end{aligned}$$

We want the second solution because it is below the length of the unstretched bungee; so the student's lowest point is 20 m below the bridge.

(b) The total time to reach the lowest point will be the time to free fall 12 m plus one-quarter of a period of the SHO. Here's the time to free fall until the bungee begins to stretch.

$$L = \tfrac{1}{2}g(\Delta t)^2 \Rightarrow \Delta t = \sqrt{\frac{2L}{g}} = \sqrt{\frac{2(12\text{ m})}{9.8\text{ m/s}^2}} = 1.56\text{ s}$$

Now find ¼ of a period for the bungee-spring motion (from unstretched to maximum stretch).

$$\frac{T}{4} = \frac{2\pi}{4}\sqrt{\frac{m}{k}} = \frac{\pi}{2}\sqrt{\frac{75\text{ kg}}{430\text{ N/m}}} = 0.66\text{ s}$$

The total time is thus 1.56 s + 0.66 s = 2.2 s.
Assess: The answers seem reasonable.

15.49. Model: Model the oscillations as SHM with no damping.

Visualize: We are given $L = 1.5$ m, $m = 0.85$ kg, $T = 2.0$ s/36, $r = 0.80$ mm, and $\Delta y = 2.5$ mm.

Solve: Use $T = 2\pi/\omega \Rightarrow \omega = 2\pi/T$ and $\omega^2 = k/m \Rightarrow k = \omega^2 m$.

$$Y = \frac{kL}{A} = \frac{(\omega^2 m)L}{\pi r^2} = \frac{\left(\left(\frac{2\pi}{T}\right)^2 m\right)L}{\pi r^2} = \frac{\left(\left(\frac{2\pi}{2.0\text{ s}/36}\right)^2(0.85\text{ kg})\right)(1.5\text{ m})}{\pi(0.80\text{ mm})^2} = 8.1\times 10^9\text{ N/m}^2$$

Assess: The answer is just a bit smaller than the values of *Y* given in the table. We did not need to know the stretch Δy which set the cord into oscillation. (Confusing Δy with ΔL in the formula for *Y* gives the wrong answer.)

15.50. Model: Model the oscillations as SHM with no damping.

Visualize: We must pay careful attention to the notation so as to not confuse the period and frequency of the mass oscillating on the spring with the period and frequency of the rotational motion. We are given $T_{\text{osc}} = 0.35$ s.

Solve: First examine the oscillatory motion.

$$T_{\text{osc}} = 2\pi\sqrt{\frac{m}{k}} \Rightarrow \frac{k}{m} = \frac{4\pi^2}{T_{\text{osc}}^2}$$

For the circular motion use Newton's second law in the radial direction. The Hooke's law force is positive since radially inward is positive in this coordinate system.

$$\Sigma F_r = k(\Delta r) = ma_r = m\omega_{\text{rot}}^2 r$$

Divide by *mr*.

$$\omega_{\text{rot}}^2 = \frac{k}{m}\frac{\Delta r}{r} \Rightarrow \omega_{\text{rot}} = \sqrt{\frac{k}{m}\frac{\Delta r}{r}} = \sqrt{\frac{4\pi^2}{T_{\text{osc}}^2}\frac{\Delta r}{r}} = \sqrt{\frac{4\pi^2}{(0.35\text{ s})^2}(0.15)} = 6.95\text{ rad/s}\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right)\left(\frac{60\text{ s}}{1\text{ min}}\right) = 66\text{ rpm}$$

Assess: This is a reasonable rotation speed.

15.51. Model: The compact car is in simple harmonic motion.
Solve: (a) The mass on each spring is $(1200\text{ kg})/4 = 300$ kg. The spring constant can be calculated as follows:

$$\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = m(2\pi f)^2 = (300\text{ kg})[2\pi(2.0\text{ Hz})]^2 = 4.74\times 10^4\text{ N/m}$$

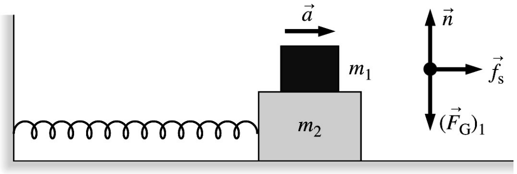
The spring constant is 4.7×10^4 N/m.
(b) The car carrying four persons means that each spring has, on the average, an additional mass of 70 kg. That is, $m = 300\text{ kg} + 70\text{ kg} = 370\text{ kg}$. Thus,

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.74\times 10^4\text{ N/m}}{370\text{ kg}}} = 1.8\text{ Hz}$$

Assess: A small frequency change from the additional mass is reasonable because frequency is inversely proportional to the square root of the mass.

15.52. Model: Assume simple harmonic motion for the two-block system without the upper block slipping. We will also use the model of static friction between the two blocks.

Visualize:



Solve: The net force on the upper block m_1 is the force of static friction due to the lower block m_2. The two blocks ride together as long as the static friction doesn't exceed its maximum possible value. The model of static friction gives the maximum force of static friction as

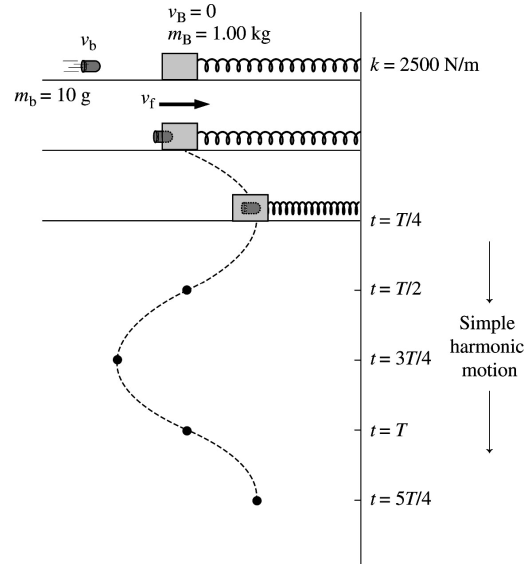
$$(f_s)_{max} = \mu_s n = \mu_s(m_1 g) = m_1 a_{max} \Rightarrow a_{max} = \mu_s g$$

$$\Rightarrow \mu_s = \frac{a_{max}}{g} = \frac{\omega^2 A_{max}}{g} = \left(\frac{2\pi}{T}\right)^2 \left(\frac{A_{max}}{g}\right) = \left(\frac{2\pi}{1.5\text{ s}}\right)^2 \left(\frac{0.40\text{ m}}{9.8\text{ m/s}^2}\right) = 0.72$$

Assess: Because the period is given, we did not need to use the block masses or the spring constant in our calculation.

15.53. Model: A completely inelastic collision between the bullet and the block resulting in simple harmonic motion.

Visualize:



Solve: **(a)** The equation for conservation of energy after the collision is

$$\frac{1}{2}kA^2 = \frac{1}{2}(m_b + m_B)v_f^2 \Rightarrow v_f = \sqrt{\frac{k}{m_b + m_B}}A = \sqrt{\frac{2500\text{ N/m}}{1.010\text{ kg}}}(0.10\text{ m}) = 5.0\text{ m/s}$$

The momentum conservation equation for the perfectly inelastic collision $p_{after} = p_{before}$ is

$$(m_b + m_B)v_f = m_b v_b + m_B v_B$$

$$(1.010\text{ kg})(5.0\text{ m/s}) = (0.010\text{ kg})v_b + (1.00\text{ kg})(0\text{ m/s}) \Rightarrow v_b = 5.0 \times 10^2\text{ m/s}$$

(b) No. The oscillation frequency $\sqrt{k/(m_b + m_B)}$ depends on the masses but not on the speeds.

15.54. Model: The DNA and cantilever undergo simple harmonic motion.

Solve: The cantilever has the same spring constant with and without the DNA molecule. The frequency of oscillation without the DNA is

$$\omega_1 = \sqrt{\frac{k}{\frac{1}{3}M}}$$

With the DNA, the frequency of oscillation is

$$\omega_2 = \sqrt{\frac{k}{\frac{1}{3}M + m}}$$

where m is the mass of the DNA.
Divide the two equations, and express $\omega_2 = \omega_1 - \Delta\omega$, where $\Delta\omega = 2\pi\Delta f = 2\pi(50\text{ Hz})$.

$$\frac{\omega_1}{\omega_2} = \frac{f_1}{f_2} = \frac{f_1}{f_1 - \Delta f} = \sqrt{\frac{\dfrac{k}{\left(\frac{1}{3}M\right)}}{\dfrac{k}{\left(\frac{1}{3}M + m\right)}}} = \sqrt{\frac{\frac{1}{3}M + m}{\frac{1}{3}M}}$$

Thus

$$f_1^2\left(\tfrac{1}{3}M\right) = (f_1 - \Delta f)^2\left(\tfrac{1}{3}M + m\right)$$

$$m = \tfrac{1}{3}M\frac{(f_1^2 - (f_1 - \Delta f)^2)}{(f_1 - \Delta f)^2} = \tfrac{1}{3}M\left(\frac{f_1^2}{(f_1 - \Delta f)^2} - 1\right)$$

Since $\Delta f \ll f_1$, (50 Hz $\ll$ 12 MHz), $(f_1 - \Delta f)^{-2} = f_1^{-2}\left(1 - \dfrac{\Delta f}{f_1}\right)^{-2} \approx f_1^{-2}\left(1 + 2\dfrac{\Delta f}{f_1}\right)$. Thus

$$m = \tfrac{1}{3}M\left(1 + 2\frac{\Delta f}{f_1} - 1\right) = \tfrac{2}{3}M\frac{\Delta f}{f_1}$$

The mass of the cantilever

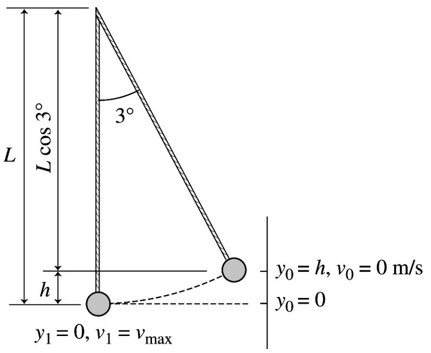
$$M = (2300\text{ kg/m}^3)(4000\times10^{-9}\text{ m})(100\times10^{-9}\text{ m}) = 3.68\times10^{-16}\text{ kg}$$

Thus the mass of the DNA molecule is

$$m = \frac{2}{3}(3.68\times10^{-16}\text{ kg})\left(\frac{50\text{ Hz}}{12\times10^{6}\text{ Hz}}\right) = 1.02\times10^{-21}\text{ kg}$$

Assess: The mass of the DNA molecule is about 6.2×10^{5} atomic mass units, which is reasonable for such a large molecule.

15.55. Model: Assume that the swinging lamp makes a small angle with the vertical so that there is simple harmonic motion.
Visualize:



Solve: **(a)** Using the formula for the period of a pendulum,

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow L = g\left(\frac{T}{2\pi}\right)^2 = (9.8\text{ m/s}^2)\left(\frac{5.5\text{ s}}{2\pi}\right)^2 = 7.5\text{ m}$$

(b) The conservation of mechanical energy equation $K_0 + U_{g0} = K_1 + U_{g1}$ for the swinging lamp is

$$\tfrac{1}{2}mv_0^2 + mgy_0 = \tfrac{1}{2}mv_1^2 + mgy_1 \Rightarrow 0\text{ J} + mgh = \tfrac{1}{2}mv_{\text{max}}^2 + 0\text{ J}$$

$$\Rightarrow v_{\text{max}} = \sqrt{2gh} = \sqrt{2g(L - L\cos 3°)}$$

$$= \sqrt{2(9.8\text{ m/s}^2)(7.5\text{ m})(1-\cos 3°)} = 0.45\text{ m/s}$$

15.56. **Model:** Assume the orangutan is a simple small-angle pendulum.
Visualize: One swing of the arm would be half a period of the oscillatory motion. The horizontal distance traveled in that time would be $2(0.90\text{ m})\sin(20°) = 0.616\text{ m}$ from analysis of a right triangle.
Solve:

$$\frac{T}{2} = \frac{2\pi}{2}\sqrt{\frac{L}{g}} = \frac{2\pi}{2}\sqrt{\frac{0.90\text{ m}}{9.8\text{ m/s}^2}} = 0.952\text{s}$$

$$\text{speed} = \frac{\text{dist}}{\text{time}} = \frac{0.616\text{ m}}{0.952\text{ s}} = 0.65\text{ m/s}$$

Assess: This isn't very fast, but isn't out of the reasonable range.

15.57. **Model:** Model swings to the left as SHM of a simple pendulum of length L (the short side) and swings to the right as SHM of a simple pendulum of length $2L$ (the long side). That is, both angles are small.

Visualize: We are given $L = 12$ cm.

Solve: (a) The total period will be half a period of the short pendulum plus half a period of the long pendulum.

$$T_{\text{tot}} = \tfrac{1}{2}T_{\text{short}} + \tfrac{1}{2}T_{\text{long}} = \frac{2\pi}{2}\sqrt{\frac{L}{g}} + \frac{2\pi}{2}\sqrt{\frac{2L}{g}} = \pi(1+\sqrt{2})\sqrt{\frac{L}{g}} = \pi(1+\sqrt{2})\sqrt{\frac{0.12\text{ m}}{9.8\text{ m/s}^2}} = 0.84\text{ s}$$

(b) To find the angle on the long side use energy conservation. The bob will swing just as high on the long side as it does on the short side. Call the rise h.

$$h = L(1-\cos\theta_{\text{short}}) = (2L)(1-\cos\theta_{\text{long}})$$

Divide by L and solve for θ_{long}.

$$\frac{1-\cos\theta_{\text{short}}}{2} = 1-\cos\theta_{\text{long}} \Rightarrow \theta_{\text{long}} = \cos^{-1}\left(\frac{1}{2} + \frac{\cos\theta_{\text{short}}}{2}\right) = \cos^{-1}\left(\frac{1}{2} + \frac{\cos 10°}{2}\right) = 7.1°$$

Assess: We expected the answer to be a bit less than 10°.

15.58. **Model:** Model the rod as a small-angle physical pendulum without damping.
Visualize: Figure 15.20 in the chapter shows that l is the distance from the pivot to the center of mass; in this case $l = L/4$.
Solve: We first need to compute the moment of inertia of a thin rod about an axis 1/4 of the way from one end. Use the parallel-axis theorem to do so:

$$I = I_{\text{c.m.}} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2 = \left(\frac{1}{12} + \frac{1}{16}\right)ML^2 = \frac{7}{48}ML^2$$

For the physical pendulum

$$f = \frac{1}{2\pi}\sqrt{\frac{Mgl}{I}} = \frac{1}{2\pi}\sqrt{\frac{Mg\left(\frac{L}{4}\right)}{\frac{7}{48}ML^2}} = \frac{1}{2\pi}\sqrt{\frac{12}{7}\frac{g}{L}} = \frac{1}{\pi}\sqrt{\frac{3}{7}\frac{g}{L}}$$

Assess: The period $T = 1/f$ is a little shorter than swinging the rod from the end, which is what we expect.

15.59. Model: Treat the lower leg as a physical pendulum.

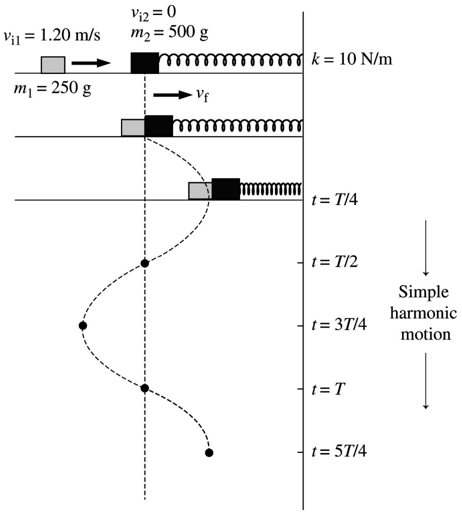
Visualize: We can determine the moment of inertia by combining $T = 2\pi\sqrt{I/mgL}$ and $T = 1/f$.

Solve: Combining the above expressions and solving for the moment of inertia we obtain

$$I = mgL/(2\pi f)^2 = (5.0\text{ kg})(9.80\text{ m/s}^2)(0.18\text{ m})/[2\pi(1.6\text{ Hz})]^2 = 8.7\times10^{-2}\text{ kg}\cdot\text{m}^2$$

Assess: NASA determines the moment of inertia of the shuttle in a similar manner. It is suspended from a heavy cable, allowed to oscillate about its vertical axis of symmetry with a very small amplitude, and from the period of oscillation one may determine the moment of inertia. This arrangement is called a torsion pendulum.

15.60. Model: A completely inelastic collision between the two gliders resulting in simple harmonic motion.
Visualize:



Let us denote the 250 g and 500 g masses as m_1 and m_2, which have initial velocities v_{i1} and v_{i2}. After m_1 collides with and sticks to m_2, the two masses move together with velocity v_f.

Solve: The momentum conservation equation $p_f = p_i$ for the completely inelastic collision is $(m_1 + m_2)v_f = m_1v_{i1} + m_2v_{i2}$. Substituting the given values,

$$(0.750\text{ kg})v_f = (0.250\text{ kg})(1.20\text{ m/s}) + (0.500\text{ kg})(0\text{ m/s}) \Rightarrow v_f = 0.400\text{ m/s}$$

We now use the conservation of mechanical energy equation:

$$(K + U_s)_{\text{compressed}} = (K + U_s)_{\text{equilibrium}} \Rightarrow 0\text{ J} + \tfrac{1}{2}kA^2 = \tfrac{1}{2}(m_1 + m_2)v_f^2 + 0\text{ J}$$

$$\Rightarrow A = \sqrt{\frac{m_1 + m_2}{k}}v_f = \sqrt{\frac{0.750\text{ kg}}{10\text{ N/m}}}(0.400\text{ m/s}) = 0.11\text{ m}$$

The period is

$$T = 2\pi\sqrt{\frac{m_1 + m_2}{k}} = 2\pi\sqrt{\frac{0.750\text{ kg}}{10\text{ N/m}}} = 1.7\text{ s}$$

15.61. Model: The block attached to the spring is oscillating in simple harmonic motion.
Solve: (a) Because the frequency of an object in simple harmonic motion is independent of the amplitude and/or the maximum velocity, the new frequency is equal to the old frequency of 2.0 Hz.

(b) The speed v_0 of the block just before it is given a blow can be obtained by using the conservation of mechanical energy equation as follows:

$$\tfrac{1}{2}kA^2 = \tfrac{1}{2}mv_{\text{max}}^2 = \tfrac{1}{2}mv_0^2$$

$$\Rightarrow v_0 = \sqrt{\frac{k}{m}}A = \omega A = (2\pi f)A = (2\pi)(2.0\text{ Hz})(0.02\text{ m}) = 0.25\text{ m/s}$$

The blow to the block provides an impulse that changes the velocity of the block:

$$J_x = F_x\Delta t = \Delta p = mv_{\text{f}} - mv_0$$

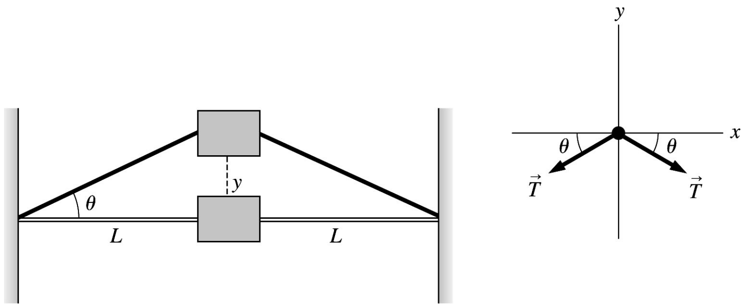
$$(-20\text{ N})(1.0\times 10^{-3}\text{ s}) = (0.200\text{ kg})v_{\text{f}} - (0.200\text{ kg})(0.25\text{ m/s}) \Rightarrow v_{\text{f}} = 0.150\text{ m/s}$$

Since v_{f} is the new maximum velocity of the block at the equilibrium position, it is equal to $A\omega$. Thus,

$$A = \frac{0.150\text{ m/s}}{\omega} = \frac{0.150\text{ m/s}}{2\pi(2.0\text{ Hz})} = 0.012\text{ m} = 1.19\text{ cm} \approx 1.2\text{ cm}$$

Assess: Because v_{f} is positive, the block continues to move to the right even after the blow.

15.62. Model: Assume the small-angle approximation.
Visualize:



Solve: The tension in the two strings pulls downward at angle θ. Thus Newton's second law is

$$\Sigma F_y = -2T\sin\theta = ma_y$$

From the geometry of the figure we can see that

$$\sin\theta = \frac{y}{\sqrt{L^2 + y^2}}$$

If the oscillation is small, then $y \ll L$ and we can approximate $\sin\theta \approx y/L$. Since y/L is $\tan\theta$, this approximation is equivalent to the small-angle approximation $\sin\theta \approx \tan\theta$ if $\theta \ll 1$ rad. With this approximation, Newton's second law becomes

$$-2T\sin\theta \approx -\frac{2T}{L}y = ma_y = m\frac{d^2y}{dt^2} \quad \Rightarrow \quad \frac{d^2y}{dt^2} = -\frac{2T}{mL}y$$

This is the equation of motion for simple harmonic motion (see Equations 15.32 and 15.46). The constants $2T/mL$ are equivalent to k/m in the spring equation or g/L in the pendulum equation. Thus the oscillation frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{2T}{mL}}$$

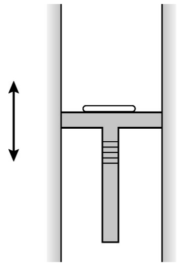
15.63. Solve: The potential energy curve of a simple harmonic oscillator is described by $U = \frac{1}{2}k(\Delta x)^2$, where $\Delta x = x - x_0$ is the displacement from equilibrium. From the graph, we see that the equilibrium bond length is $x_0 = 0.13$ nm. We can find the bond's spring constant by reading the value of the potential energy U at a displacement Δx and using the potential energy formula to calculate k.

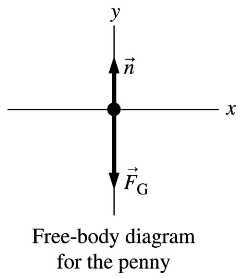
x (nm)	Δx (nm)	U (J)	k (N/m)
0.11	0.02	0.8×10^{-19} J	400
0.10	0.03	1.9×10^{-19} J	422
0.09	0.04	3.4×10^{-19} J	425

The three values of k are all very similar, as they should be, with an average value of 416 N/m. Knowing the spring constant, we can now calculate the oscillation frequency of a hydrogen atom on this "spring" to be

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{416 \text{ N/m}}{1.67\times10^{-27} \text{ kg}}} = 7.9\times10^{13} \text{ Hz}$$

15.64. Visualize:





Solve: (a) Newton's second law applied to the penny along the y-axis is

$$F_{\text{net}} = n - mg = ma_y$$

$\vec{F}_{\text{net}}$ is upward at the bottom of the cycle (positive a_y), so $n > mg$. The speed is maximum when passing through equilibrium, but $a_y = 0$ so $n = mg$. The critical point is the *highest* point. $\vec{F}_{\text{net}}$ points down and a_y is negative. If a_y becomes sufficiently negative, n drops to zero and the penny is no longer in contact with the surface.

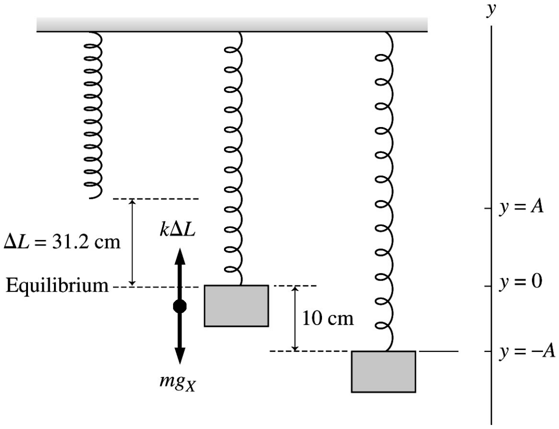
(b) When the penny loses contact ($n = 0$), the equation for Newton's law becomes $a_{\text{max}} = g$. For simple harmonic motion,

$$a_{\text{max}} = A\omega^2 \Rightarrow \omega = \sqrt{\frac{g}{A}} = \sqrt{\frac{9.8 \text{ m/s}^2}{0.040 \text{ m}}} = 15.65 \text{ rad}$$

$$\Rightarrow f = \frac{\omega}{2\pi} = \frac{15.65 \text{ rad/s}}{2\pi} = 2.5 \text{ Hz}$$

15.65. Model: The vertical oscillations constitute simple harmonic motion.

Visualize:



Solve: At the equilibrium position, the net force on mass m on Planet X is:

$$F_{\text{net}} = k\Delta L - mg_X = 0\ \text{N} \Rightarrow \frac{k}{m} = \frac{g_X}{\Delta L}$$

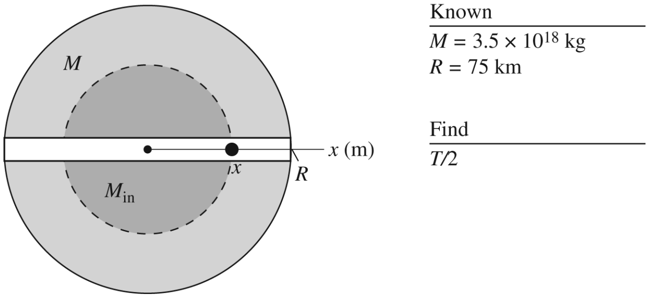
For simple harmonic motion $k/m = \omega^2$, thus

$$\omega^2 = \frac{g_X}{\Delta L} \Rightarrow \omega = \sqrt{\frac{g_X}{\Delta L}} = \frac{2\pi}{T} \Rightarrow g_X = \left(\frac{2\pi}{T}\right)^2 \Delta L = \left(\frac{2\pi}{14.5\ \text{s}/10}\right)^2 (0.312\ \text{m}) = 5.86\ \text{m/s}^2$$

Assess: Because ordinary tasks seemed easier than on earth we expected an answer less than $9.8\ \text{m/s}^2$.

15.66. Model: Model the object as a sphere of uniform density. The missing mass from the tunnel is negligible.

Visualize: Put the origin of the coordinate system at the center of the spherical object.



Solve: (a) Use Newton's law of gravitation with M_{in} the mass with $r \le x$.

$$F = G\frac{mM_{\text{in}}}{x^2}$$

To find M_{in} assume the ratio of masses equals the ratio of volumes since the density is uniform.

$$\frac{M_{\text{in}}}{M} = \frac{V_{\text{in}}}{V} = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi R^3} \Rightarrow M_{\text{in}} = M\frac{x^3}{R^3}$$

Now plug into the previous equation.

$$F = G\frac{mM_{\text{in}}}{x^2} = G\frac{mM}{x^2}\frac{x^3}{R^3} = G\frac{mM}{R^3}x$$

This force is linear in x and represents a Hooke's law restoring force. The collection of constants in front of the x is the k for this SHM; i.e., $k = GmM/R^3$.

(b) The time it takes the astronaut to emerge on the other side of the asteroid is one-half of a period of the SHM.

$$\frac{T}{2} = \frac{2\pi}{2}\sqrt{\frac{m}{k}} = \pi\sqrt{\frac{m}{GmM/R^3}} = \pi\sqrt{\frac{R^3}{GM}}$$

$$= \pi\sqrt{\frac{(75\text{ km})^3}{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(3.5\times10^{18}\text{ kg})}} = 4200\text{ s} = 70\text{ min}$$

Assess: This is the same amount of time for a satellite in orbit just above the asteroid's surface to make half an orbit. Notice that we did not need the mass of the astronaut (or the particle), so it is irrelevant.

15.67. Model: The doll's head is in simple harmonic motion and is damped.
Solve: (a) The oscillation frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow k = m(2\pi f)^2 = (0.015\text{ kg})(2\pi)^2(4.0\text{ Hz})^2 = 9.475\text{ N/m}$$

The spring constant is 9.5 N/m.
(b) Using $A(t) = A_0 e^{-bt/2m}$, we get

$$(0.5\text{ cm}) = (2.0\text{ cm})e^{-b(4.0\text{ s})/(2\times0.015\text{ kg})} \Rightarrow 0.25 = e^{-(133.3\text{ s/kg})b}$$
$$\Rightarrow -(133.33\text{ s/kg})b = \ln 0.25 \Rightarrow b = 0.0104\text{ kg/s} \approx 0.010\text{ kg/s}$$

15.68. Model: The oscillator is in simple harmonic motion.
Solve: The maximum displacement at time t of a damped oscillator is

$$x_{\max}(t) = Ae^{-t/2\tau} \Rightarrow -\frac{t}{2\tau} = \ln\left(\frac{x_{\max}(t)}{A}\right)$$

Using $x_{\max} = 0.98A$ at $t = 0.50$ s, we can find the time constant τ to be

$$\tau = -\frac{0.50\text{ s}}{2\ln(0.98)} = 12.375\text{ s}$$

25 oscillations will be completed at $t = 25T = 12.5$ s. At that time, the amplitude will be

$$x_{\max,\,12.5\text{ s}} = (10\text{ cm})e^{-12.5\text{ s}/(2)(12.375\text{ s})} = 6.0\text{ cm}$$

15.69. Model: The vertical oscillations are damped and follow simple harmonic motion.
Solve: The position of the ball is given by $x(t) = Ae^{-(t/2\tau)}\cos(\omega t + \phi_0)$. The amplitude $A(t) = Ae^{-(t/2\tau)}$ is a function of time. The angular frequency is

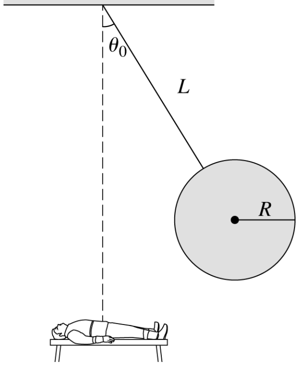
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{(15.0\text{ N/m})}{0.500\text{ kg}}} = 5.477\text{ rad/s} \Rightarrow T = \frac{2\pi}{\omega} = 1.147\text{ s}$$

Because the ball's amplitude decreases to 3.0 cm from 6.0 cm after 30 oscillations, that is, after $30\times1.147\text{ s} = 34.41\text{ s}$, we have

$$3.0\text{ cm} = (6.0\text{ cm})e^{-(34.414\text{ s}/2\tau)} \Rightarrow 0.50 = e^{-(34.41\text{ s}/2\tau)} \Rightarrow \ln(0.50) = \frac{-34.41\text{ s}}{2\tau} \Rightarrow \tau = 25\text{ s}$$

15.70. Model: Model the swinging disk as a physical (not simple) pendulum.

Visualize:



Known

$L = 6.0$ m
$R = 1.0$ m
$\theta_0 = 10°$
$\theta_{25\text{ min}} = 7°$
$v_y = 1.0$ mm/s
$\Delta y = 3.0$ m

Find

$v_{\text{final pass}}$

Solve: For the physical pendulum we need the moment of inertia. The massless rod contributes nothing. The moment of inertia of the disk about the point where the rod is attached at the top (by the parallel axis theorem) is

$$I_{\text{disk}} = I_{\text{c.m.}} + m(L+R)^2 = \tfrac{1}{2}mR^2 + m(L+R)^2$$

Now we compute the undamped frequency of oscillation.

$$\omega_0 = \sqrt{\frac{mg(L+R)}{I}} = \sqrt{\frac{mg(L+R)}{\frac{1}{2}mR^2 + m(L+R)^2}} = \sqrt{\frac{g(L+R)}{\frac{1}{2}R^2 + (L+R)^2}} = \sqrt{\frac{(9.8\text{ m/s}^2)(7.0\text{ m})}{\frac{1}{2}(1.0\text{ m})^2 + (7.0\text{ m})^2}} = 1.177\text{ rad/s}$$

The actual damped angular frequency is

$$\omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}}$$

so we need to figure out τ from the information given about the damping from $10°$ to $7°$ in 1500 s.

$$(0.70)A_0 = A_0 e^{-t/2\tau} \Rightarrow \ln(0.70) = -t/2\tau \Rightarrow \tau = -\frac{t}{2\ln(0.70)} = -\frac{1500\text{ s}}{2\ln(0.70)} = 2103\text{ s}$$

Plug this into the equation for ω.

$$\omega = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}} = \sqrt{(1.177\text{ rad/s})^2 - \frac{1}{4(2103\text{ s})^2}} = 1.177\text{ s}$$

That is, $\omega \approx \omega_0$, so this is a lightly damped system. Now we turn to the speed of the bottom edge of the blade, which is $(L+2R)$ from the pivot point. The equation for the angular motion of the damped oscillator is

$$\theta(t) = A_0 e^{-t/2\tau}\cos(\omega t)$$

Find the derivative of this by the product rule.

$$\frac{d\theta}{dt} = A_0\left(\frac{-1}{2\tau}e^{-t/2\tau}\cos\omega t - e^{-t/2\tau}\omega\sin\omega t\right)$$

Now multiply by the distance from the pivot to get the linear speed.

$$v=(L+2R)\frac{d\theta}{dt}=(L+2R)A_0\left(\frac{-1}{2\tau}e^{-t/2\tau}\cos\omega t-e^{-t/2\tau}\omega\sin\omega t\right)$$
$$=(L+2R)(-A_0e^{-t/2\tau})\left(\frac{1}{2\tau}\cos\omega t+\omega\sin\omega t\right)$$

From the speed of the lowering pendulum we compute the time until Bond is freed.

$$t=\frac{3.0\text{ m}}{1.0\text{ mm/s}}-30\text{ s}=2970\text{ s}$$

The amplitude at that time is

$$A_{\text{final}}=A_0e^{-t/2\tau}=(10°)e^{-(2970\text{ s})/2(2103\text{ s})}=4.935°$$

The pendulum passes the lowest point when $\cos\omega t=0$. When $\cos\omega t=0$ then $\sin\omega t=1$. Put this all together to get the speed of the last pass at the bottom before Bond is freed. Drop the minus sign since we want the speed.

$$v=(L+2R)A_{\text{final}}(0+\omega\cdot 1)=(6.0\text{ m}+2.0\text{ m})\left(4.935°\left(\frac{\pi\text{ rad}}{180°}\right)\right)(1.177\text{ rad/s})=0.81\text{ m/s}$$

Assess: The answers seem reasonable. We did not need the mass of the circular blade since it canceled in the formula for ω.

15.71. Model: The motion is a damped oscillation.

Solve: The position of the air-track glider is $x(t)=Ae^{-(t/2\tau)}\cos(\omega t+\phi_0)$, where $\tau=m/b$ and

$$\omega=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

Using $A=0.20$ m, $\phi_0=0$ rad, and $b=0.015$ kg/s,

$$\omega=\sqrt{\frac{4.0\text{ N/m}}{0.250\text{ kg}}-\frac{(0.015\text{ kg/s})^2}{4(0.250\text{ kg})^2}}=\sqrt{16-9\times10^{-4}}\text{ rad/s}=4.0\text{ rad/s}$$

Thus the period is

$$T=\frac{2\pi}{\omega}=\frac{2\pi\text{ rad}}{4.0\text{ rad/s}}=1.57\text{ s}$$

The amplitude at $t=0$ s is $x_0=A$ and the amplitude will be equal to $e^{-1}A$ at a time given by

$$\frac{1}{e}A=Ae^{-(t/2\tau)}\Rightarrow t=2\tau=2\frac{m}{b}=33.3\text{ s}$$

The number of oscillations in a time of 33.3 s is $(33.3\text{ s})/(1.57\text{ s})=21$.

15.72. Model: The oscillator is in simple harmonic motion.

Solve: The maximum displacement, or amplitude, of a damped oscillator decreases as $x_{\max}(t)=Ae^{-t/2\tau}$, where τ is the time constant. We know $x_{\max}/A=0.60$ at $t=50$ s, so we can find τ as follows:

$$-\frac{t}{2\tau}=\ln\left(\frac{x_{\max}(t)}{A}\right)\Rightarrow\tau=-\frac{50\text{ s}}{2\ln(0.60)}=48.9\text{ s}$$

Now we can find the time t_{30} at which $x_{\max}/A=0.30$:

$$t_{30}=-2\tau\ln\left(\frac{x_{\max}(t)}{A}\right)=-2(48.9\text{ s})\ln(0.30)=118\text{ s}$$

The undamped oscillator has a frequency $f = 2\text{ Hz} = 2$ oscillations per second. Damping changes the oscillation frequency slightly, but the text notes that the change is negligible for "light damping." Damping by air, which allows the oscillations to continue for well over 100 s, is certainly light damping, so we will use $f = 2.0$ Hz. Then the number of oscillations before the spring decays to 30% of its initial amplitude is

$$N = f \cdot t_{30} = (2\text{ oscillations/s}) \cdot (118\text{ s}) = 236\text{ oscillations}$$

15.73. Solve: The solution of the equation

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

is $x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0)$. The first and second derivatives of $x(t)$ are

$$\frac{dx}{dt} = -\frac{Ab}{2m}e^{-bt/2m}\cos(\omega t + \phi_0) - A\omega e^{-bt/2m}\sin(\omega t + \phi_0)$$

$$\frac{d^2x}{dt^2} = \left[\left(\frac{Ab^2}{4m^2} - A\omega^2\right)\cos(\omega t + \phi_0) + \frac{\omega Ab}{m}\sin(\omega t + \phi_0)\right]e^{-bt/2m}$$

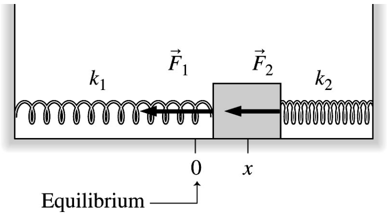
Substituting these expressions into the differential equation, the terms involving $\sin(\omega t + \phi_0)$ cancel and we obtain the simplified result

$$\left(\frac{-b^2}{4m^2} - \omega^2 + \frac{k}{m}\right)\cos(\omega t + \phi_0) = 0$$

Because $\cos(\omega t + \phi_0)$ is not equal to zero in general,

$$\frac{-b^2}{4m^2} - \omega^2 + \frac{k}{m} = 0 \Rightarrow \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

15.74. Model: The two springs obey Hooke's law.
Visualize:



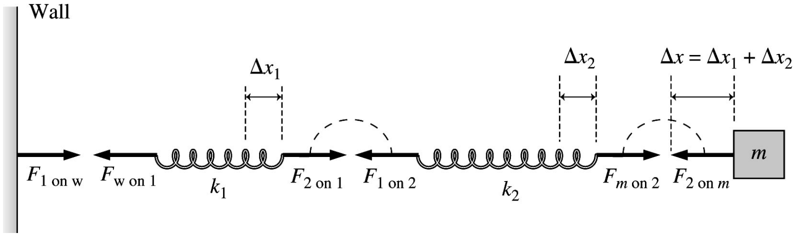
Solve: There are *two* restoring forces on the block. If the block's displacement x is positive, *both* restoring forces—one pushing, the other pulling—are directed to the left and have negative values:

$$(F_{\text{net}})_x = (F_{\text{sp 1}})_x + (F_{\text{sp 2}})_x = -k_1x - k_2x = -(k_1 + k_2)x = -k_{\text{eff}}x$$

where $k_{\text{eff}} = k_1 + k_2$ is the effective spring constant. This means the oscillatory motion of the block under the influence of the two springs will be the *same* as if the block were attached to a single spring with spring constant k_{eff}. The frequency of the blocks, therefore, is

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{k_1}{4\pi^2 m} + \frac{k_2}{4\pi^2 m}} = \sqrt{f_1^2 + f_2^2}$$

15.75. Model: The two springs obey Hooke's law. Assume massless springs.
Visualize: Each spring is shown separately. Note that $\Delta x = \Delta x_1 + \Delta x_2$.



Solve: Only spring 2 touches the mass, so the net force on the mass is $F_m = F_{2\text{ on }m}$. Newton's third law tells us that $F_{2\text{ on }m} = F_{m\text{ on }2}$ and that $F_{2\text{ on }1} = F_{1\text{ on }2}$. From $F_{\text{net}} = ma$, the net force on a massless spring is zero. Thus $F_{\text{w on }1} = F_{2\text{ on }1} = k_1\Delta x_1$ and $F_{m\text{ on }2} = F_{1\text{ on }2} = k_2\Delta x_2$. Combining these pieces of information,

$$F_m = k_1\Delta x_1 = k_2\Delta x_2$$

The net displacement of the mass is $\Delta x = \Delta x_1 + \Delta x_2$, so

$$\Delta x = \Delta x_1 + \Delta x_2 = \frac{F_m}{k_1} + \frac{F_m}{k_2} = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)F_m = \frac{k_1 + k_2}{k_1 k_2}F_m$$

Turning this around, the net force on the mass is

$$F_m = \frac{k_1 k_2}{k_1 + k_2}\Delta x = k_{\text{eff}}\Delta x \quad \text{where } k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2}$$

k_{eff}, the proportionality constant between the force on the mass and the mass's displacement, is the effective spring constant. Thus the mass's angular frequency of oscillation is

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m}\frac{k_1 k_2}{k_1 + k_2}}$$

Using $\omega_1^2 = k_1/m$ and $\omega_2^2 = k_2/m$ for the angular frequencies of either spring acting alone on *m*, we have

$$\omega = \sqrt{\frac{(k_1/m)(k_2/m)}{(k_1/m) + (k_2/m)}} = \sqrt{\frac{\omega_1^2\omega_2^2}{\omega_1^2 + \omega_2^2}}$$

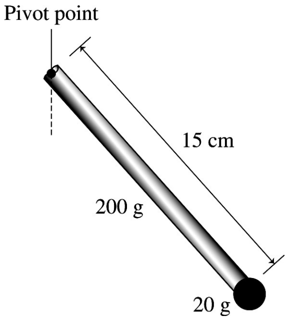
Since the actual frequency *f* is simply a multiple of ω, this same relationship holds for f:

$$f = \sqrt{\frac{f_1^2 f_2^2}{f_1^2 + f_2^2}}$$

Challenge Problems

15.76. Model: The rod is thin and uniform with moment of inertia described in Table 12.2. The clay ball is a particle located at the end of the rod. The ball and rod together form a physical pendulum. The oscillations are small.

Visualize:



Solve: The moment of inertia of the composite pendulum formed by the rod and clay ball is

$$I_{\text{rod}+\text{ball}} = I_{\text{rod}} + I_{\text{ball}} = \frac{1}{3}m_{\text{rod}}L^2 + m_{\text{ball}}L^2$$

$$= \frac{1}{3}(0.200 \text{ kg})(0.15 \text{ m})^2 + (0.020 \text{ kg})(0.15 \text{ m})^2 = 1.95\times10^{-3} \text{ kg m}^2$$

The center of mass of the rod and ball is located at a distance from the pivot point of

$$y_{\text{cm}} = \frac{(0.200 \text{ kg})\left(\dfrac{0.15}{2} \text{ m}\right) + (0.020 \text{ kg})(0.15 \text{ m})}{(0.200 \text{ kg} + 0.020 \text{ kg})} = 8.18\times10^{-2} \text{ m}$$

The frequency of oscillation of a physical pendulum is

$$f = \frac{1}{2\pi}\sqrt{\frac{Mgl}{I}} = \frac{1}{2\pi}\sqrt{\frac{(0.220 \text{ kg})(9.8 \text{ m/s}^2)(8.18\times10^{-2} \text{ m})}{1.95\times10^{-3} \text{ kg m}^2}} = 1.51 \text{ Hz}$$

The period of oscillation $T = \dfrac{1}{f} = 0.66$ s.

15.77. Model: Model the sphere as a small-angle physical pendulum without damping.
Visualize: Figure 15.20 in the chapter shows that l is the distance from the pivot to the center of mass; in this case $l = R$.
Solve: We first need to compute the moment of inertia of a sphere about an axis tangent to the edge. Use the parallel-axis theorem to do so:

$$I = I_{\text{c.m.}} + Md^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

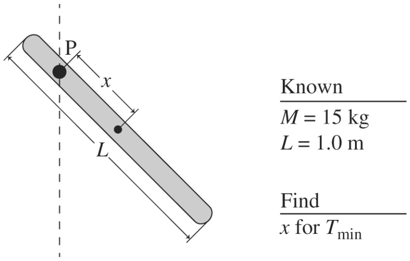
For the physical pendulum

$$f = \frac{1}{2\pi}\sqrt{\frac{Mgl}{I}} = \frac{1}{2\pi}\sqrt{\frac{MgR}{\frac{7}{5}MR^2}} = \frac{1}{2\pi}\sqrt{\frac{5}{7}\frac{g}{R}}$$

Assess: The frequency is less than for a simple pendulum of length R as we expect.

15.78. Model: The rod is a physical pendulum whose moment of inertia about the middle is $I_{\text{c.m.}} = \frac{1}{12}ML^2$.

Visualize:



Solve: (a) The period of a physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{Mgx}}$$

where x is the distance from the pivot to the center of mass. We use the parallel axis theorem to find I.

$$I_{\text{rod}} = I_{\text{c.m.}} + Mx^2 = \tfrac{1}{12}ML^2 + Mx^2$$

Plug this in to the formula for the period and cancel M.

$$T = 2\pi\sqrt{\frac{\frac{1}{12}ML^2 + Mx^2}{Mgx}} = 2\pi\sqrt{\frac{\frac{1}{12}L^2 + x^2}{gx}} = 2\pi\left(\frac{\frac{1}{12}L^2 + x^2}{gx}\right)^{1/2}$$

This is the period as a function of L and x. To find the value of x for which the period is a minimum we take the derivative with respect to x and eventually set it equal to zero. To take the derivative we first use the exponent rule and then the quotient rule on what is inside.

$$\frac{dT}{dx} = 2\pi\left(\frac{1}{2}\right)\left(\frac{\frac{1}{12}L^2 + x^2}{gx}\right)^{-1/2}\left(\frac{gx(2x) - \left(\frac{1}{12}L^2 + x^2\right)g}{g^2x^2}\right) = \pi\left(\frac{\frac{1}{12}L^2 + x^2}{gx}\right)^{-1/2}\left(\frac{2x^2 - \left(\frac{1}{12}L^2 + x^2\right)}{gx^2}\right)$$

We could attempt to simplify this further before setting equal to zero, but we can see that the only way the entire expression can be zero is if the numerator of the last factor is zero. This will save lots of time and effort.

$$2x^2 - \left(\tfrac{1}{12}L^2 + x^2\right) = 0 \Rightarrow 2x^2 = \tfrac{1}{12}L^2 + x^2 \Rightarrow x = \frac{1}{\sqrt{12}}L$$

(b) Go back to the formula for the period and put in the value $L/\sqrt{12}$ for x and plug in the numbers.

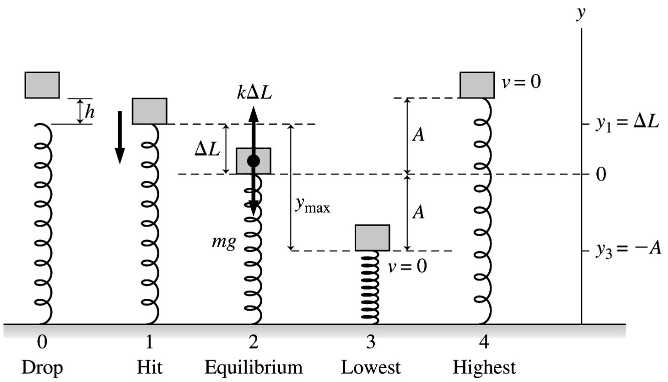
$$T = 2\pi\sqrt{\frac{\frac{1}{12}L^2 + x^2}{gx}} = 2\pi\sqrt{\frac{\frac{1}{12}L^2 + \left(\frac{1}{\sqrt{12}}L\right)^2}{g\left(\frac{1}{\sqrt{12}}L\right)}} = 2\pi\sqrt{\frac{\frac{1}{12}L^2 + \frac{1}{12}L^2}{g\left(\frac{1}{\sqrt{12}}L\right)}}$$

$$= 2\pi\sqrt{\frac{\frac{1}{6}L^2}{g\left(\frac{1}{\sqrt{12}}L\right)}} = 2\pi\sqrt{\frac{\frac{\sqrt{12}}{6}L}{g}} = 2\pi\sqrt{\frac{\frac{\sqrt{3}}{3}L}{g}} = 2\pi\sqrt{\frac{\frac{\sqrt{3}}{3}(1.0\text{ m})}{(9.8\text{ m/s}^2)}} = 1.5\text{ s}$$

Assess: This is a reasonable period. Notice that the mass canceled out so we did not need to know it.

15.79. Model: The block undergoes simple harmonic motion after sticking to the spring. Energy is conserved throughout the motion.

Visualize:



It's essential to carefully visualize the motion. At the highest point of the oscillation the spring is *stretched* upward.

Solve: We've placed the origin of the coordinate system at the equilibrium position, where the block would sit on the spring at rest. The spring is compressed by ΔL at this point. Balancing the forces requires $k\Delta L = mg$. The angular frequency is $w^2 = k/m = g/\Delta L,$ so we can find the oscillation frequency by finding ΔL. The block hits the spring (1) with kinetic energy. At the lowest point (3), kinetic energy and gravitational potential energy have been transformed into the spring's elastic energy. Equate the energies at these points:

$$K_1 + U_{1g} = U_{3s} + U_{3g} \Rightarrow \tfrac{1}{2}mv_1^2 + mg\Delta L = \tfrac{1}{2}k(\Delta L + A)^2 + mg(-A)$$

We've used $y_1 = \Delta L$ as the block hits and $y_3 = -A$ at the bottom. The spring has been compressed by $\Delta y = \Delta L + A$. Speed v_1 is the speed after falling distance h, which from free-fall kinematics is $v_1^2 = 2gh.$ Substitute this expression for v_1^2 and $mg/\Delta L$ for k, giving

$$mgh + mg\Delta L = \frac{mg}{2(\Delta L)}(\Delta L + A)^2 + mg(-A)$$

The mg term cancels, and the equation can be rearranged into the quadratic equation

$$(\Delta L)^2 + 2h(\Delta L) - A^2 = 0$$

The positive solution is

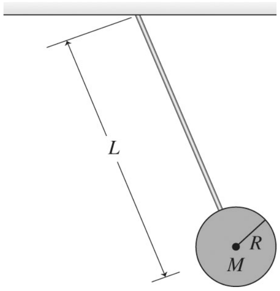
$$\Delta L = \sqrt{h^2 + A^2} - h = \sqrt{(0.030\text{ m})^2 + (0.100\text{ m})^2} - 0.030\text{ m} = 0.0744\text{ m}$$

Now that ΔL is known, we can find

$$\omega = \sqrt{\frac{g}{\Delta L}} = \sqrt{\frac{9.80\text{ m/s}^2}{0.0744\text{ m}}} = 11.48\text{ rad/s} \Rightarrow f = \frac{\omega}{2\pi} = 1.8\text{ Hz}$$

15.80. Model: Assume that the pendulum is restricted to small angles so $\sin\theta \approx \theta.$

Visualize: Because the string is massless the center of mass is at the center of the sphere, so $l = L$ in the formula for a physical pendulum: $T = 2\pi\sqrt{\frac{I}{Mgl}}$.



Solve:
(a) First we use the parallel axis theorem to find I for the sphere around the pivot point at the top (L away): $I = I_{\text{cm}} + Md^2 = \frac{2}{5}MR^2 + ML^2$.

$$T = 2\pi\sqrt{\frac{I}{Mgl}} = 2\pi\sqrt{\frac{\frac{2}{5}MR^2 + ML^2}{MgL}} = 2\pi\sqrt{\frac{\frac{2}{5}R^2 + L^2}{gL}}$$

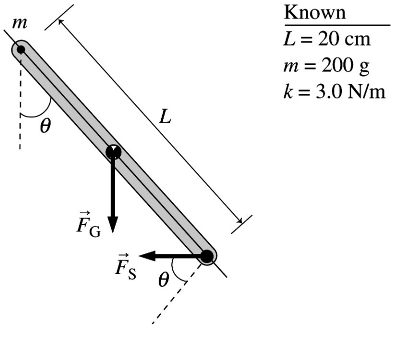
This reduces to the traditional simple pendulum formula $T = 2\pi\sqrt{\frac{L}{g}}$ in the limit $L \gg R$.

(b)

$$\frac{T_{\text{real}}}{T_{\text{simple}}} = \frac{2\pi\sqrt{\dfrac{\frac{2}{5}R^2 + L^2}{gL}}}{2\pi\sqrt{\frac{L}{g}}} = \frac{\sqrt{\frac{2}{5}R^2 + L^2}}{L} = \frac{\sqrt{\frac{2}{5}(0.010\text{ m})^2 + (1.0\text{ m})^2}}{1.0\text{ m}} = 1.00002$$

Assess: Making the simple pendulum approximation introduces an error of only 0.002% with these numbers.

15.81. Model: The rod is thin and uniform.
Visualize:



Solve: We must derive our own equation for this combination of a pendulum and spring. For small oscillations, $\vec{F}_s$ remains horizontal. The net torque around the pivot point is

$$\tau_{\text{net}} = I\alpha = -F_s L\cos\theta - F_{\text{G}}\left(\frac{L}{2}\right)\sin\theta$$

With $\alpha = \dfrac{d^2\theta}{dt^2}$, $F_{\text{G}} = mg$, $F_{\text{s}} = k\Delta x = kL\sin\theta$, and $I = \dfrac{1}{3}mL^2$,

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{m}\sin\theta\cos\theta - \frac{3g}{2L}\sin\theta$$

We can use $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta.$ For small angles, $\sin\theta \approx \theta$ and $\sin 2\theta \approx 2\theta.$ So

$$\frac{d^2\theta}{dt^2} = -\left(\frac{3k}{m} + \frac{3g}{2L}\right)\theta$$

This is the same as Equations 15.32 and 15.46 with

$$\omega = \sqrt{\frac{3k}{m} + \frac{3g}{2L}}$$

The frequency of oscillation is thus

$$f = \frac{1}{2\pi}\sqrt{\frac{3(3.0\text{ N/m})}{(0.200\text{ kg})} + \frac{3(9.8\text{ m/s}^2)}{2(0.20\text{ m})}} = 1.73\text{ Hz}$$

The period $T = \frac{1}{f} = 0.58\text{ s}.$

Assess: Fewer than two oscillations per second is reasonable. The rod’s angle from the vertical must be small enough that $\sin 2\theta \approx 2\theta.$ This is more restrictive than other examples, which only require that $\sin\theta \approx \theta.$